Parametrization of the Kobayashi-Maskawa Matrix

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The quark mixing matrix (Kobayashi-Maskawa matrix) is expanded in powers of a small parameter λ equal to $\sin\theta_c = 0.22$. The term of order λ^2 is determined from the recently measured B lifetime. Two remaining parameters, including the CP-nonconservation effects, enter only the term of order λ^3 and are poorly constrained. A significant reduction in the limit on ϵ'/ϵ possible in an ongoing experiment would tightly constrain the CP-nonconservation parameter and could rule out the hypothesis that the only source of CP nonconservation is the Kobayashi-Maskawa mechanism.

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The quark mixing of the weak-interaction current in the standard model is described by the 3×3 Kobayashi-Maskawa (KM) matrix[1]

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \qquad (1)$$

The element V_{us} is quite well determined to be equal to 0.22. This and other information suggest that V differs from unity by a small quantity. Here we set

$$0.22 = V_{us} = \lambda \qquad (2)$$

and consider an expansion of V in powers of λ. A recent measurement of the lifetime τ_B of B particles yields the result[2]

$$V_{cb} \approx 0.06. \qquad (3)$$

This suggests to us that V_{cb} is of order λ^2 rather than λ so that we set

$$V_{cb} = A\lambda^2$$

with $A \approx \frac{5}{4}$. To order λ^2 the KM matrix can then be written

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & 0 \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ 0 & -A\lambda^2 & 1 \end{pmatrix}.$$

We now want to go to order λ^3. Unitarity then prescribes the following form:

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & \lambda^2 A \\ \lambda^3 A(1-\rho-i\eta) & -\lambda^2 A & 1 \end{pmatrix}, \qquad (4)$$

where two new parameters ρ and η must be introduced. Equation (4) can be derived from the standard KM form by assuming that s_2 and s_3 are of order λ^2 and making the replacements[3]

$$\lambda = s_1, \qquad (5a)$$

$$A\lambda^2 = (s_2^2 + s_3^2 + 2s_2 s_3 \cos\delta)^{1/2}, \qquad (5b)$$

$$A^2\lambda^4\eta = s_2 s_3 \sin\delta, \qquad (5c)$$

$$A\lambda^2(\rho^2+\eta^2)^{1/2} = s_3, \qquad (5d)$$

or

$$A\lambda^2[(1-\rho)^2+\eta^2]^{1/2} = s_2. \qquad (5e)$$

Only three of the equations (5b)–(5e) are independent. The phase convention has been changed from the standard form so that CP nonconservation enters only terms of order λ^3.

Given the values of λ and A we look for empirical constraints on ρ and η. If we neglect CP nonconservation for the moment, terms of the order λ^4 (which enter along the diagonal and in the $\lambda^2 A$ terms) are too small to be of importance given experimental and theoretical uncertainties. Therefore the simple form (4) is adequate for present analyses. The only significant constraint now comes from the limit on the ratio of $b\to u$ to $b\to c$ transitions which yields[4]

$$|V_{ub}/V_{cb}| < 0.2 \qquad (6)$$

or

$$\rho^2 + \eta^2 < 1. \qquad (7)$$

From Eq. (7) it follows that

$$V_{td} < 2A\lambda^3. \qquad (8)$$

I note in passing that Eqs. (6)–(8) show the consistency of the expansion in powers of λ because they limit the coefficients of the λ^3 terms. Many other experimental constraints on the KM matrix are discussed in the literature.[1] When we neglect CP nonconservation all of these are now of no significance. For example, an upper limit on

$\mathrm{Re}(V_{td}V_{ts})$ may be set based on the box-diagram contribution[5] to $K_L \to \mu^+\mu^-$. For $m_t \sim 40$ GeV this limit[6] is about 0.02; a stronger limit[7] comes from Eq. (8):

$$|V_{td}V_{ts}| < 2\lambda^5 A^2 \approx 2\times 10^{-3}.$$

All CP-nonconserving effects are proportional to $-\mathrm{Im}\,V_{td} = -\mathrm{Im}\,V_{ub} = \lambda^3 A\eta \le 1.5\times 10^{-2}$ where the last inequality follows from Eq. (7). For the K^0 system CP-nonconserving effects depend on $V_{td} \times V_{ts}$; because $V_{ts} \sim \lambda^2$ whereas $V_{us} \sim \lambda$ the characteristic CP-nonconserving parameter is

$$\lambda^4 A^2 \eta = s_2 s_3 \sin\delta \le 4\times 10^{-3}. \tag{9}$$

Since $V_{td}V_{ts}$ is of order λ^5 it is necessary for CP-nonconservation calculation to expand the imaginary part of the KM matrix to order λ^5. By a particular phase convention we keep V_{ud}, V_{us}, V_{cd}, V_{ts}, and V_{tb} real. Unitarity then requires the form

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta + i\eta\frac{1}{2}\lambda^2) \\ -\lambda & 1-\frac{1}{2}\lambda^2 - i\eta A^2\lambda^4 & A\lambda^2(1+i\eta\lambda^2) \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}, \tag{10}$$

where we have demanded that the imaginary part of the unitarity relation be satisfied to order λ^5 and the real part only to order λ^3. The term $i\eta A\lambda^5/2$ in V_{ub}, which could be transferred to V_{td}, is needed for unitarity but can be neglected in present calculations. The two new terms $i\eta A\lambda^4$ in V_{cs} and V_{cb} represent the leading CP-nonconserving pieces of these elements and cannot be neglected in general.

As first emphasized by Gilman and Wise,[8] as a result of penguin diagrams a nonzero value of the CP-nonconserving parameter ϵ' is expected from the KM matrix. Following Hagelin and Gilman[6] one finds

$$|\epsilon'| \approx 0.02\, A^2\lambda^4\eta \approx 0.03\eta|\epsilon|, \tag{11}$$

where the empirical value of ϵ is used for the last equality and the coefficient (0.02 or 0.03) has an uncertainty of at least a factor of 2 associated with the hadronic matrix element. The present limit ($|\epsilon'/\epsilon| < 0.02$) provides at best only a modest constraint on η. Howver, if this limit is reduced by a factor of 5, as is possible in an ongoing experiment,[9] the upper limit on $|\eta|$ would be reduced to the order of 0.1. Since the calculation involves an evaluation of ϵ' and not of the ratio ϵ'/ϵ, this constraint holds even if CP nonconservation is not entirely due to the KM mechanism; for example, ϵ might result from a combination of a superweak and a KM contribution. Of course, if the ϵ' due to the KM mechanism is approximately cancelled by some other source of CP nonconservation this constraint would not be correct.

As has been commented by many people,[10] the measured B lifetime has the consequence that if CP nonconservation is to be explained entirely by the KM mechanism the value of the CP-nonconserving parameter ($A^2\lambda^4\eta = s_2 s_3 \sin\delta$) must be close to the upper limit of Eq. (9) if m_t is not too large. Thus for $m_t \lesssim 40$ GeV a value of $\eta \gtrsim 0.5$ is needed.[11] From Eq. (11) this would correspond to a value of ϵ'/ϵ close to the present upper limit. If the present limit is reduced by a factor of 5 and if m_t is found to be of the order 40 GeV or less it would be very difficult to explain the value of ϵ using the KM mechanism alone. The KM matrix may in this case still be one of the sources of CP nonconservation and as discussed above it may be a major source of a nonzero value for ϵ'.

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[1] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973). For a recent review see L. L. Chau, Phys. Rep. 95, 1 (1983).

[2] I use the value $\tau_B = 1.2\times 10^{-12}$ sec reported by the Mark II group at the storage ring PEP in Proceedings of the International Lepton-Photon Conference, Cornell University, August 1983 (to be published), together with Eq. (6a) from P. H. Ginsparg, S. L. Glashow, and M. B. Wise, Phys. Rev. Lett. 50, 1415 (1983), based on the work of Cortes *et al.*, Phys. Rev. D 25, 188 (1982). A somewhat better procedure is to use the theoretical formula for $b \to ce\nu$ together with τ_B and the experimental branching ratio; within the present uncertainties (probably 20% experimental and 20% theoretical on the number 0.06) the two procedures

give the same result.

[3]My notation is more closely related to that of L. Maiani, in *Proceedings of the International Symposium on Lepton and Photon Interaction at High Energies, Hamburg, 1977* (DESY, Hamburg, 1977), p. 867. His parameters are given by $s_\theta = \lambda$, $s_\gamma = \lambda^2 A$, and $s_\beta e^{i\delta} = \lambda^3 A(\rho - i\eta)$.

[4]B. Gittelman and P. Franzini, J. Phys. (Paris), Colloq. 43, C3-110 (1982).

[5]R. E. Shrock and M. B. Voloshin, Phys. Lett. 87B, 375 (1979).

[6]J. S. Hagelin and F. Gilman, SLAC Report No. SLAC PUB-3087, 1983 (to be published).

[7]For values of m_t much larger than m_W, box-diagram constraints may become significant. I am indebted to J. S. Hagelin for this point.

[8]F. J. Gilman and M. B. Wise, Phys. Lett. 83B, 83 (1979).

[9]B. Winstein, in Proceedings of the Symposium on Intense Medium Energy Sources of Strangeness, Santa Cruz, California, March 1983 (to be published).

[10]See, for example, P. H. Ginsparg *et al.*, in Ref. 3.

[11]I have used Eq. (10) of Ref. 6. This has an uncertainty of at least a factor of 2.