Eur. Phys. J. A **40**, 299–312 (2009)
DOI 10.1140/epja/i2009-10803-9

THE EUROPEAN PHYSICAL JOURNAL A

Regular Article – Theoretical Physics

Power laws in elementary and heavy-ion collisions[⋆]

A story of fluctuations and nonextensivity?

G. Wilk[1,a] and Z. Włodarczyk[2,b]

[1] The Andrzej Sołtan Institute for Nuclear Studies, Hoża 69; 00-689 Warsaw, Poland
[2] Institute of Physics, Jan Kochanowski University, Świętokrzyska 15, 25-406 Kielce, Poland

Received: 31 January 2009
Published online: 23 May 2009 –
Communicated by U.-G. Meißner

Abstract. We review from the point of view of nonextensive statistics the ubiquitous presence in elementary and heavy-ion collisions of power law distributions. Special emphasis is placed on the conjecture that this is just a reflection of some intrinsic fluctuations existing in the hadronic systems considered. These systems are summarily described by a single parameter q playing the role of a nonextensivity measure in the nonextensive statistical models based on Tsallis entropy.

PACS. 89.75.-k Complex systems – 24.60.-k Statistical theory and fluctuations – 25.75.Dw Particle and resonance production – 25.75.-q Relativistic heavy-ion collisions

1 Introduction

In many domains of physics, especially in elementary and heavy-ion collisions, for decades the prevailing understanding was that the exponential shape of most of the observed spectra of produced secondaries suggests their statistical (or even thermodynamical) origin. Therefore, when looking for spectra of transverse momenta it is commonly assumed that the observed inverse mean transverse momenta (characterizing the widths of such exponential spectra) play the role of temperature of the hadronizing system [1]. This assumption allows us to use, in what follows, the whole machinery of statistical models. This is specially important in the case of high-energy collisions of heavy ions because it allows us to use the tools of statistical physics to investigate the possibility of the formation in such collisions of a new state of matter, the so-called *Quark Gluon Plasma* (QGP) [2] (in this case one is interested in details of *hadron* $\Longleftrightarrow$ *QGP* phase transition, which can only be investigated in this way).

However, one should always keep in mind another possibility, namely that such behavior can just be due to the fact that, when presenting our data, out of many particles produced (tens or thousands at present, say N to be specific) we select only *one* to make the corresponding plots. Single-particle distributions averaged over many events are what is usually published. But this means that the remaining $(N-1)$ particles will act as a kind of *heath bath*. Assuming that this heath bath is homogenous and large it is natural to expect that its action can be described by a single parameter, which we then call temperature, T, and identify with the temperature encountered in statistical models [3]. What one apparently observes then is just the usual Boltzmann-Gibbs (BG) statistics at work.

The above reasoning assumes that any dynamics of the set of remaining $(N-1)$ particles is mostly averaged and what results looks very much like a state of hadronic matter remaining in *thermal equilibrium* characterized by a temperature T. Some effects, however, can survive this *equilibration* process and can show up as apparent departures from the assumed thermal equilibrium. This is usually regarded as a departure from BG statistics and, finally, considered as a kind of failure of the simple statistical approach. Such observations are therefore a subject of separate investigations in which many different (dynamical) ideas compete (like, for example, the production of resonances and the flow of matter, to name only two) [4]. However, because there is only a limited amount of available data, usually one cannot decide which of the proposed dynamical remedies is right or, if we agree that they should all be present, in what proportion they show up[1].

[⋆] This paper is part of the Topical Issue *Statistical Power Law Tails in High-Energy Phenomena*.

[a] e-mail: `wilk@fuw.edu.pl`
[b] e-mail: `wlod@pu.kielce.pl`

[1] This is connected with the important problem of how much information given measurements are providing us with. This is, so far, only sporadically discussed in the domain of high-energy multiparticle production processes using information theory approach in which the entropies mentioned here are regarded as measures of information [5,6]. This subject is, however, beyond the scope of our review.

On the other hand, one can argue that, perhaps, it is the form of statistical model used which should be modified in such a way as to account (at least to some extent) for detected irregularities (*i.e.*, for departures from the BG approach). Therefore, instead of inventing and investigating different dynamical assumptions, one can instead investigate the possibility of replacing the usual statistical model based on BG entropy, S_{BG}, by its modified version based on some other form of entropy discussed in the literature [7]. Such models are widely known nowadays from other branches of physics and are used whenever a physical system under investigation shows memory effects of any kind, experiences long-range correlations (*i.e.*, is in a sense "small" because its size is comparable with the range of forces acting in it), experiences some intrinsic fluctuations, or the phase space in which it operates is limited or has fractal structure. In this review we shall discuss the application of such a model taken in the form proposed by Tsallis, *i.e.*, in the form based on the so-called Tsallis entropy, $S_T = S_q$ [8]:

$$S_q = \frac{(1 - \sum_i p_i^q)}{q-1} \quad \overset{q\to 1}{\Longrightarrow} \quad S_{BG} = -\sum_i p_i \ln p_i. \quad (1)$$

Notice that $S_{BG} = S_{q=1}$. The S_q is nonextensive because for any two independent systems A and B (in the usual sense, *i.e.*, for which $p_{ij}(A+B) = p_i(A)p_j(B)$), one observes that

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B). \quad (2)$$

In this sense the entropic index q is a measure of the nonextensivity in the system without, however, directly showing up its cause. This must be provided from elsewhere.

To shed more light on the physics involved here, let us come back to the previous reasoning with some effective thermal bath being formed by the $(N-1)$ particles remaining after the selection of the one used for making final histograms. Notice that, in high-energy multiparticle production reactions observed in elementary and heavy-ion collisions we are interested in here, such thermal baths usually (*i.e.*, after more detailed scrutiny) do not satisfy conditions allowing us to introduce the notion of thermal equilibrium in the BG sense: they are always finite and can be hardly considered as being homogenous (in fact, in many cases they occupy only a fraction of the allowed phase space [9] or even have a fractal-like structure [10] and it is known that usually the hadronizing system under consideration experiences long-range correlations). This means therefore that such a heath bath cannot be described by a single parameter T. The simplest thing to do seems to be to allow for some fluctuations of the parameter T and to replace it by its mean value $T \to T_0 = \langle T\rangle$ and by one more parameter describing its fluctuations, using, for example, the (normalized) variance [11,12]:

$$\omega = \frac{\langle \left(\frac{1}{T}\right)^2\rangle - \langle \frac{1}{T}\rangle^2}{\langle \frac{1}{T}\rangle^2}\,. \quad (3)$$

In this approach

$$q = 1 + \omega. \quad (4)$$

It can be next shown that such a heath bath leads in a natural way to the following q-exponential distribution (called also Tsallis distributions)[2]:

$$\exp\left(-\frac{X}{\lambda}\right) \Rightarrow \exp_q\left(-\frac{X}{\lambda}\right) = \left[1-(1-q)\frac{X}{\lambda}\right]^{\frac{1}{1-q}}. \quad (5)$$

This is the power law we were searching for in different reactions [6,13,15–22], the results of which will be reviewed here[3].

The physical picture presented above can be made more formal by saying that one replaces here the notion of strict *local thermal equilibrium*, customarily assumed in all applications of statistical models, by the notion of some kind of *stationary state*, which is being formed in the collision and which already includes some interactions. This concept can be introduced in different ways. For example, in [29] it was a random distortion of energy and momentum conservation caused by the surrounding system which resulted in the emergence of some nonextensive equilibrium. In [30,31] the two-body energy composition in transport theory formulation of the collision process is replaced by a generalized energy sum, $h(E_1,E_2)$, which is assumed to be associative but which is not necessarily a simple addition and contains contributions stemming from pair interaction (in the simplest case). It turns out that under quite general assumptions about the function h, a division of the total energy among free particles is possible. Different forms of the function h then lead to different forms of entropy formula, among which one encounters the known Tsallis form. The origin of this kind of thinking can be traced back to the analysis of the q-Hagedorn model proposed some time ago in [32].

We close this section with a historical note. The recognition that some, apparently unexpected power law distributions can be due to fluctuations came to us from the observation in cosmic ray physics [33] that there exists a *long flying component* (LFC) phenomenon in the propagation of the initial flux of incoming nucleons. For example, instead of the normally expected exponential fall off of the depth distribution of the starting points of cascades, z,

$$\frac{\mathrm{d}N(z)}{\mathrm{d}z} = \mathrm{const}\cdot\exp\left(-z/\lambda\right), \quad (6)$$

one rather observes an $\exp_q(-z/\lambda)$ distribution, *i.e.*, a Tsallis-like power law behavior given by eq. (5) with z replacing X (here $\lambda \sim 1/\sigma$ is the mean free path describing

[2] Except for [12] the discussion concerning the meaning of the parameter q was limited to the case of $q > 1$ only. As already mentioned in [12], the case of $q < 1$ seems not so much connected with any genuine fluctuations but rather, in some way, with limitations of the allowed phase space [13]. It is worth mentioning that the idea of possible fluctuations of otherwise intensive quantities has already been formalized by introducing the new concept of the so-called *superstatistics* [14].

[3] It should be noticed that there are also some other investigations on this subject [23–28], with rather similar conclusions, which we will not discuss here.

the propagation of the incoming flux in the atmosphere with σ being the relevant cross-section). On the other hand, some time ago we have shown in [34] that this effect can be explained by assuming that hadronic cross-sections should be regarded as fluctuating quantities with widths (defined as normalized dispersion), $\omega_\sigma = \langle\sigma^2\rangle/\langle\sigma\rangle^2 - 1$ (and growing logaritmically with energy). We were at that time prompted by the fact that such an idea was widely investigated in the usual hadronic collisions [35][4]. It was then quite natural to connect the parameter q with fluctuations. This was done in [11,12], as mentioned above.

We shall review our results in this field in the next section. Section 3 contains some new recent developments in this field. The final section contains our conclusions and a summary.

2 Review of fluctuations in multiparticle production processes

High-energy collisions result in a multitude of particles of different kinds being produced. Most are just mesons of all kinds (overwhelmingly pions). For those who are looking for some new and/or rare phenomena they form unwanted background which must somehow be substracted, for others they are a subject of thorough investigations allowing us to look inside the very early stages of the collision process as well as at the hadronization stage of the matter produced (proceeding probably via the formation of the QGP, for example). In both cases a simple and trustworthy representation of data is very important, this justifies our investigations in this field to be reported here.

2.1 Generalized heat bath; fluctuations of temperature

We first recall the physical picture behind the generalized heat bath introduced in sect. 1 which we have proposed in [11,12]. Our reasoning was as follows. Suppose we have a thermodynamic system, in a small (mentally separated) part of which the temperature can take different values, *i.e.*, in the whole system it fluctuates with $\Delta T \sim T$. Let $\xi(t)$ describes stochastic changes of temperature in time. If the mean temperature of the system temperature is $\langle T\rangle = T_0$ then, as a result of fluctuations in some small selected region, the actual temperature T' equals

$$T' = T_0 - b\,\xi(t)\,T, \tag{7}$$

where the constant b is defined by the actual definition of the stochastic process under consideration, *i.e.*, by $\xi(t)$, which is assumed to satisfy the condition that

$$\langle\xi(t)\rangle = 0 \tag{8}$$

and whose correlator, $\langle\xi(t)\,\xi(t+\Delta t)\rangle$, for sufficiently fast changes is equal to

$$\langle\xi(t)\,\xi(t+\Delta t)\rangle = 2\,D\,\delta(\Delta t). \tag{9}$$

[4] It is interesting to notice that this idea has been revitalized very recently in [36] and connected with the fluctuations in the gluonic content of hadrons.

The inevitable exchange of heat between any selected region of the system and the rest leads to the equilibration of the temperature in the whole system. The corresponding process of heat conductance is described by the following equation [37]:

$$c_p\,\rho\,\frac{\partial T}{\partial t} - a\,(T' - T) = 0, \tag{10}$$

where c_p, ρ and a are, respectively, the specific heat under constant pressure, density and the coefficient of external conductance. Using T' as defined in (7) we finally get the linear differential equation for the temperature T with $\tau = b = \frac{c_p\rho}{a}$:

$$\frac{\partial T}{\partial t} + \left[\frac{1}{\tau} + \xi(t)\right]T = \frac{1}{\tau}T_0. \tag{11}$$

It can be now shown that this equation leads to the Langevin equation with multiplicative noise term resulting in fluctuations of the temperature T given in the form of a gamma function [11]

$$f(T) = \frac{1}{\Gamma(\alpha)}\,\mu\left(\frac{\mu}{T}\right)^{\alpha-1}\exp\left(-\frac{\mu}{T}\right) \tag{12}$$

and characterized by the parameters μ and α,

$$\mu = \frac{\phi}{D} \qquad \text{and} \qquad \alpha = \frac{1}{q-1} = \frac{1}{\tau\,D}\,. \tag{13}$$

This is to be compared with eq. (3) in which now $\omega = \tau D$. Function $f(T)$ as given by eq. (12) is the distribution that should be used to smear the parameter T in the usual exponential distribution of the BG statistical model and which results in the Tsallis distribution, eq. (5). To summarize: a small addition of the multiplicative noise described by a damping constant in the Langevin equation results in a stationary distribution of particle momenta, which develops a power law tail at high values[5].

2.2 Transverse and longitudinal dynamics

To begin our presentation we first set the stage. The characteristic pattern of the multiparticle production processes is that most of the secondaries are produced with small transverse momenta p_T (mostly below 1 GeV) and are therefore concentrated in the longitudinal phase space given by the longitudinal momenta p_L (which is described in terms of the rapidity $y = \frac{1}{2}\ln\frac{E+p_L}{E-p_L}$, where $E = \sqrt{m_T^2 + p_L^2}$ with $m_T = \sqrt{m^2 + p_T^2}$ being the so-called transverse mass (m is mass of the particle); in other notation $E = m_T\cosh y$ and $p_L = m_T\sinh y$). The terms *transverse* and *longitudinal* are defined with respect to the direction of the colliding particles. Data are presented

[5] Actually, as shown in [14] when discussing *superstatistics*, there is a whole class of functions leading from $\exp(X)$ to $\exp_q(X)$. But only this has a simple physical interpretation as presented here. More general version of Langevin equation containing also additive noise have been considered in [38].

as distributions either in p_T or in y. In both cases they show exponential behavior either in p_T or in the energy $E = \langle m_T \rangle \cosh y$ (with $\langle m_T \rangle = \sqrt{m^2 + \langle p_T \rangle^2}$):

$$\frac{\mathrm{d}N}{\mathrm{d}p_T} = C_{p_T} p_T \exp\left(-\frac{p_T}{T}\right); \quad \frac{\mathrm{d}N}{\mathrm{d}y} = C_y \exp\left(-\frac{E}{T}\right). \quad (14)$$

One observes dramatic differences in both distributions reflected by the differences in the values of the parameter T, which is of the order of one hundred MeV in p_T space (where $T = T_{p_T}$ and is universal, *i.e.*, essentially energy independent) and tens of GeV (depending on the energy of collision) in p_L (or y) space (where $T = T_{p_L}$ and depends on energy). This means that the two distributions reflect different physics: those in p_T space are believed to be essentially "thermal-like" and subject to a thermodynamic interpretation whereas, those in p_L space are sensitive to the available energy and to the multiplicity of produced secondaries. Because of this their fluctuation patterns will be different, *i.e.*, when described by Tsallis power-like form eq. (5) the corresponding parameters $(q_T - 1)$ and $(q_L - 1)$ will differ dramatically. Also the physical meaning of these parameters will be different reflecting different sources of fluctuations.

2.3 Longitudinal phase space

We start with the longitudinal distribution in rapidity (averaged over p_T). In fig. 1 one observes that $q < 1$ is effectively cutting off the allowed longitudinal phase space (here defined by the initial available energy $M = 100$ GeV and assumed constant transverse mass $m_T = 0.44$ GeV and weakly depending on the assumed multiplicity of the produced particles N). Actually, from eq. (5) it is obvious that only such combinations of q and X and λ are allowed for which $[1 - (1-q)X/\lambda] > 0$. In [13], when fitting longitudinal distributions without restricting the available energy by introducing the so-called inelasticity coefficient $K < 1$, the only role of q, which was found to be $q < 1$ there, was to limit the amount of energy used (showing the necessity of introducing inelasticity when considering multiparticle production processes, cf., [39] for review on this subject). For $q > 1$ one observes a visible enhancement of distribution tails.

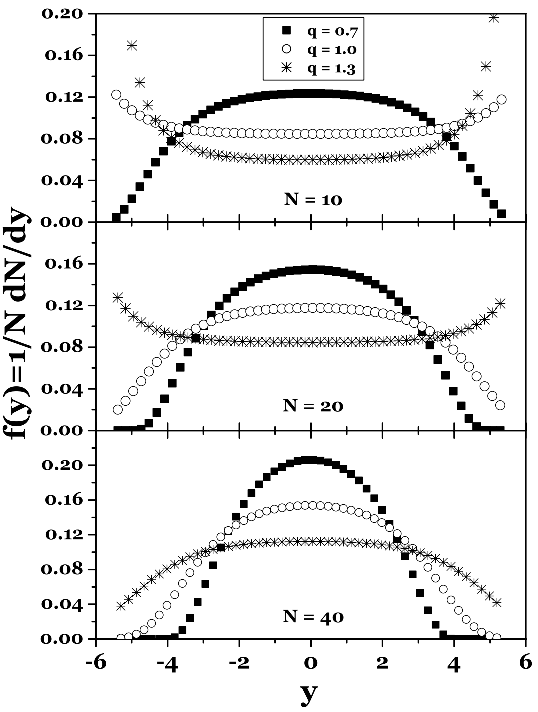

Fig. 1. Examples of the most probable rapidity distributions as given by eq. (14) for hadronizing mass $M = 100$ GeV decaying into N secondaries of (transverse) mass $m_T = 0.4$ GeV each for different values of the parameter $q = Q_L$ (reproduced from [18] with kind permission of Springer-Verlag).

The examples of fits to the actually observed single-particle distributions in rapidity are shown in figs. 2 and 4. In the left panel of fig. 2 (see [19] for details) results for pp and $p\bar{p}$ collisions at energies varying between $\sqrt{s} = 20$ GeV to 1800 GeV are displayed. From each listed energy of collision, $E_{cm} = \sqrt{s}$, only a fraction K_q has been used for the production of secondaries (according to [39]). The other input was the mean multiplicity of charged secondaries produced in nonsingle diffractive reactions at given energy: $\bar{n}_{ch} = -7.0 + 7.2 s^{0.127}$ [1] (corresponding to the total number of produced particles, $N = \frac{3}{2}\bar{n}_{ch}$). The allowed phase space is one dimensional with only a small energy dependence of the mean transverse momentum allowed, $\langle p_T \rangle = 0.3 + 0.044 \ln(\sqrt{s}/20)$ [1] (all secondaries will be assumed to be pions of mass $\mu = 0.14$ GeV).

As discussed in detail in [19], one gets in this case not only the parameter q but also the true inelasticity, K, of the reaction (in fact, even, for the first time, its distribu-

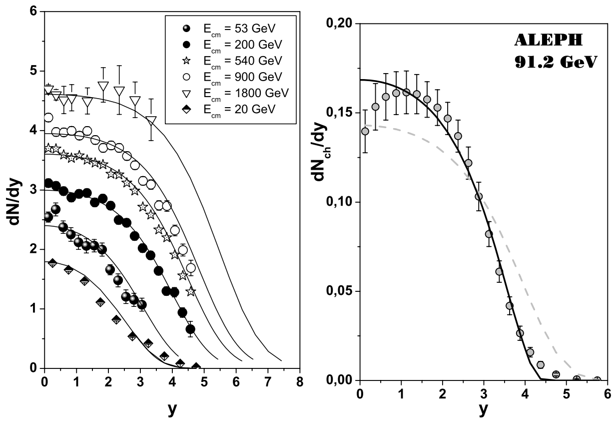

Fig. 2. Examples of applying the nonextensive approach to longitudinal distributions. Left panel: fit to rapidity spectra for charged pions produced in pp and $\bar{p}p$ collisions at different energies [40]. Right panel: rapidity spectra measured in e^+e^- annihilations at 91.2 GeV [41] (the dashed line is for $K_q = 1$ and $q = 1$, whereas the full line is our fit with $K_q = 1$ and $q = 0.6$). (Reprinted from F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, *Information theory in high-energy physics (extensive and nonextensive approach)*, Physica A **344**, 568 (2004), with kind permission of Elsevier, `http://www.elsevier.com`.)

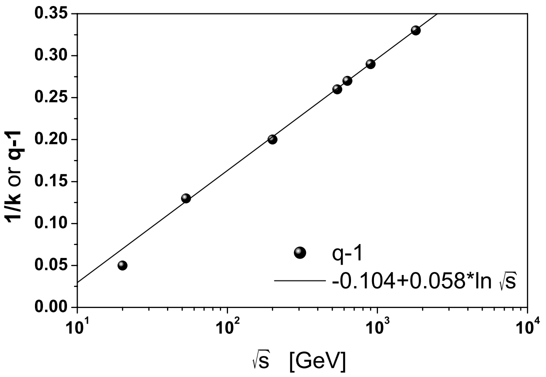

Fig. 3. The values of the nonextensivity parameter q obtained in fits shown in the left panel of fig. 2 compared with the values of the parameter k of a negative binomial distribution fit to the corresponding multiplicity distributions as given in [1]. (Reprinted fig. 6 from F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Phys. Rev. D **67**, 114002 (2003) URL: `http://link.aps.org/abstract/PRD/v67/e114002`, DOI: 10.1103/PhysRevD.67.114002, with kind permission of the American Physical Society.)

tion, $\chi(K)$). Let us, however, concentrate on the parameter q, which bears information on fluctuations. It turns out to be energy dependent as presented in fig. 3. Surprisingly enough it turned out that the same behavior is observed for the inverse of k characterizing the so-called Negative Binomial distribution (NB) [19] of the multiplicity of observed secondaries, which depends on two parameters: the mean multiplicity $\langle n_{ch}\rangle$ and the parameter k ($k \geq 1$) affecting its width; $\sigma(n_{ch})$ is the dispersion,

$$\frac{1}{k} = \frac{\sigma^2(n_{ch})}{\langle n_{ch}\rangle^2} - \frac{1}{\langle n_{ch}\rangle}\,. \tag{15}$$

For $k \to 1$ NB approaches a geometrical distribution whereas for $k^{-1} \to 0$ it approaches a Poissonian distribution. In general it is found [19] that $\frac{1}{k} = -0.104 + 0.058 \cdot \ln\sqrt{s}$, which fits the obtained values very nicely, cf. fig. 3.

To fully understand the possible physical meaning of the parameter $q(= q_L)$ in this case let us remind ourselves that, in general, the nonextensivity parameter q summarizes the action of several factors, each of which leads to a deviation from the simple form of the extensive BG statistics, as was mentioned before, out of which we are interested most in the possible intrinsic fluctuations existing in the hadronizing system [11]. Notice that in our fits we have not explicitly accounted for the fact that each event has its own multiplicity, N, but we have used only its mean value, $\langle N\rangle$, as given by experiment, where $\langle N\rangle = \sum NP(N)$ with $P(N)$ being the multiplicity distribution[6]. On the other hand, the parameter T in this case is not so much a temperature, but only a kind of "partition temperature", understood as mean energy per produced particle, *i.e.*, $T \sim W/\langle N\rangle$ (where $W = K\sqrt{s}$, where $\sqrt{s}$ is the total energy of collision) [42]. Therefore in this case one can just as well speak about the fluctuations of $\langle N\rangle$. Following therefore the ideas of [11] we would like to draw attention to the fact that the value of k^{-1} may also be understood as the measure of fluctuations of the mean multiplicity (for example, in the usual Poissonian multiplicity distribution characterized just by a single parameter, the constant mean multiplicity $\bar{n}$), and in the case when such fluctuations are given by a gamma distribution with normalized variance $D(\bar{n})$, one obtains the Negative Binomial multiplicity distribution with

$$\frac{1}{k} = D(\bar{n}) = \frac{\sigma^2(\bar{n})}{\langle\bar{n}\rangle^2}\,. \tag{16}$$

This is because in this case one has [43]:

$$P(n) = \int_0^\infty \mathrm{d}\bar{n}\,\frac{e^{-\bar{n}}\bar{n}^n}{n!}\cdot\frac{\gamma^k\bar{n}^{k-1}e^{-\gamma\bar{n}}}{\Gamma(k)} = \frac{\Gamma(k+n)}{\Gamma(1+n)\Gamma(k)}\cdot\frac{\gamma^k}{(\gamma+1)^{k+n}}\,, \tag{17}$$

where $\gamma = \frac{k}{\langle\bar{n}\rangle}$.

Therefore the situation in longitudinal phase space is the following: when there are only statistical fluctuations in the hadronizing system one expects a Poissonian form of the corresponding multiplicity distributions. The existence of intrinsic (dynamical) fluctuations means that one allows the mean multiplicity $\bar{n}$ to fluctuate. It is natural to assume that these fluctuations contribute predominantly to the longitudinal phase space, *i.e.*, that $D(\bar{n}) = q - 1$ and that

$$q = 1 + \frac{1}{k}\,. \tag{18}$$

This is observed in the data.

The right-hand panel of fig. 2 displays results for e^+e^- annihilations for which, by definition, $K_q = 1$ (because always all the energy of initial leptons is available for the production of secondaries) and which can be fitted *only* with $q < 1$ (in our case $q = 0.6$). This should be contrasted with the results obtained describing the p_T distributions instead, where one finds $q > 1$ [23]. This point deserves closer scrutiny. The result for $q = 1$ clearly shows that the observed discrepancies are not connected with the particular value of q, but rather with some additional mechanisms operating here, the action of which would, however, change our results only slightly (for example, a possibility of two rather than one source or y-dependent $\langle p_T\rangle$, as mentioned already in [6]). With the above reservations, let us then take a closer look at the possible origin of $q < 1$. We have already encountered a similar situation when in [13] $q < 1$ was simply closing the allowed *a priori* phase space, acting therefore as inelasticity parameter K. When considered as a signal of fluctuations (similar to the $q > 1$ case) [12] it causes trouble because in this case the temperature T does not reach an equilibrium state, in fact one now has that the source term (right-hand side of eq. (11)) is $T_0/\tau - (q-1)E/\tau$ rather than T_0/τ used

[6] Actually, we have used only its charged part, $\langle n_{ch}\rangle$, assuming that $N = \frac{3}{2}\langle n_{ch}\rangle$, *i.e.*, neglecting in addition also possible fluctuations between the number of charged and neutral secondaries.

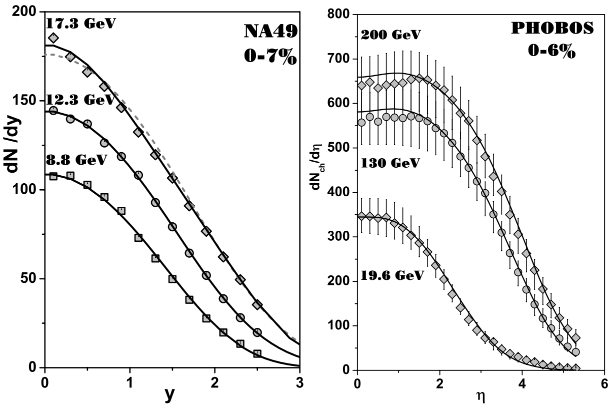

Fig. 4. Examples of applying a nonextensive approach to longitudinal distributions. Left panel: fits to NA49 data for Pb+Pb collisions [44]. Right panel: fits to PHOBOS data for Au+Au collisions [45]. (Left panel: reprinted from F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, *Information theory approach (extensive and nonextensive) to high-energy multiparticle production processes*, Physica A **340**, 467 (2004); right panel: from F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, *Information theory in high-energy physics (extensive and nonextensive approach)*, Physica A **344**, 568 (2004); both with kind permission of Elsevier, `http://www.elsevier.com`.)

for $q > 1$ case (cf. [12]). This means than that in this case we have a kind of dissipative transfer of energy from the region where (due to fluctuations) the temperature T is higher. It could be any kind of convection-type flow of energy; for example, it could be connected with the emission of particles from this region (for example, in our case from a quark (q) and antiquark ($\bar{q}$) jets formed in the first $e^+e^- \to q + \bar{q}$ to gluons and $q\bar{q}$ pairs and later on to finally observed hadrons). This means that $q < 1$ signals that in the reaction considered, where $K_q = 1$ and where we have to account for the whole energy exactly, conservation laws start to be important and there is no possibility for a stationary state with constant final temperature to develop. Instead, the temperature T depends on the energy[7], and for large energies tends to zero (notice that in this case one has a limitation on the allowed energy of the produced secondaries: $E \leq T_0/(1-q)$). This is not the case for the p_T distribution analysis [23] because most p_T are small and are not influenced by the conservation laws but instead reflect a kind of stationary state with $q > 1$.

Now look at the left-hand panel of fig. 4. It shows fits to NA49 data [44] on π^- production in PbPb collisions at three different energies per nucleon. The obtained values of nonextensivities and the corresponding inelasticities $(q; K_q = 3 \cdot K_q^{\pi^-})$ are: (1.2; 0.33) for 17.3 GeV, (1.164; 0.3) for 12.3 GeV and (1.04; 0.22) for 8.6 GeV. The origin of $q > 1$ in this case is not yet clear. The inelasticity seems to grow with energy. It is also obvious that, for higher energies, some new mechanism starts to operate because we cannot obtain agreement with data using only energy conservation. The best fit for 17.3 GeV for NA49 data actually for the case of $q = 1$ and two sources separated in rapidity by $\Delta y = 0.83$ (cf. [6] for other details).

Finally, the right-hand panel of fig. 4 presents fits to pion production in Au+Au collisions [45] for the most central events (covering collisions proceeding with an impact parameter range 0–6%)[8]. They can be fitted by choosing $K_q = 1$ and then $q = 1.29$, 1.26 and 1.27 for energies 19.6, 130 and 200 GeV, respectively (cf. [20] for other details). As before, the origin of $q > 1$ in this case is not yet clear.

Although the situation in AA collisions is not yet clear, we are quite confident that the interpretation of the q parameter offered here remains valid. But, before settling this, one point has to be addressed. Namely, the above q were in fact q_L responsible for the longitudinal dynamics only. On the other hand, multiplicity distributions are sensitive to $p = \sqrt{p_L^2 + p_T^2}$ and, as we have seen here, both p_L and p_T show traces of fluctuations by leading to $q > 1$. However, as we shall see below, $(q_T - 1) \ll (q_L - 1)$ (what fits nicely the fact that the p_T space is very limited in comparison to the p_L one). Because there are no data measuring p_T distributions at all values of rapidity y, *i.e.*, providing correlations between parameters $(T; q) = (T_L; q_L)$ for longitudinal momenta (rapidity) distributions and $(T; q) = (T_T; q_T)$ for transverse momenta distributions, we offer only the following approximate answer. Noticing that $q - 1 = \sigma^2(T)/T^2$ (*i.e.*, it is given by fluctuations of total temperature T) and assuming that $\sigma^2(T) = \sigma^2(T_L) + \sigma^2(T_T)$, one can estimate that the resulting values of q should not be too different from

$$q = \frac{q_L\, T_L^2 + q_T\, T_T^2}{T^2} - \frac{T_L^2 + T_T^2}{T^2} + 1, \qquad (19)$$

which, for $T_L \gg T_T$, as is in our case, leads to the result that $q \sim q_L$, *i.e.*, it is given by the longitudinal (rapidity) distributions only.

2.4 Transverse phase space

As discussed before, the transverse phase space seems to be mainly dominated by the thermodynamical-like effects governed by the temperature T [1]. It is therefore the best place too look for any fluctuations of temperature, *i.e.*, to look for any deviation of the inverse slope of transverse momenta distributions, $\mathrm{d}N/\mathrm{d}p_T$, from an exponential shape. That such deviations are really observed is seen

[7] Actually, in the case considered in [12] fluctuations depend on energy in the same way leaving the relative variance ω constant and leading to $q = 1 - \omega$.

[8] Actually these data are originally presented not for the rapidity y defined by the energy E and the longitudinal momentum p_L but for the so-called pseudorapidity η defined by the total momentum p and the longitudinal momentum p_L instead. There is therefore some ambiguity when transferring them from η to y because of the poor knowledge of the rapidity dependence of the mean transverse momentum needed for such operation.

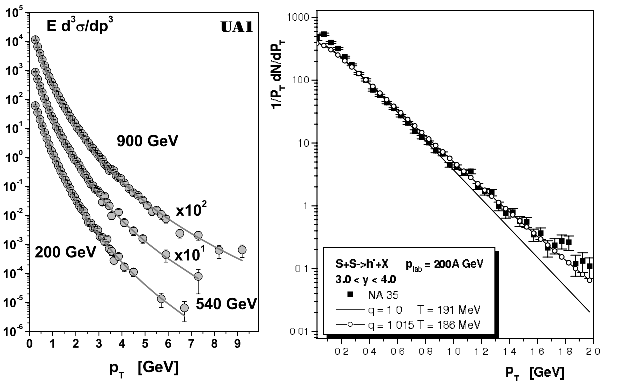

Fig. 5. Examples of applying a nonextensive approach to transverse momenta distributions. Left panel: fits to p_T spectra from the $p\bar{p}$ UA1 experiment [46] for different energies (see text for details) (reprinted from F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, *Information theory approach (extensive and nonextensive) to high-energy multiparticle production processes*, Physica A **340**, 467 (2004), with kind permission of Elsevier, `http://www.elsevier.com.`). Right panel: fits to $S+S$ data from [47] (reproduced with kind permission of IOP Publishing Ltd from [15]).

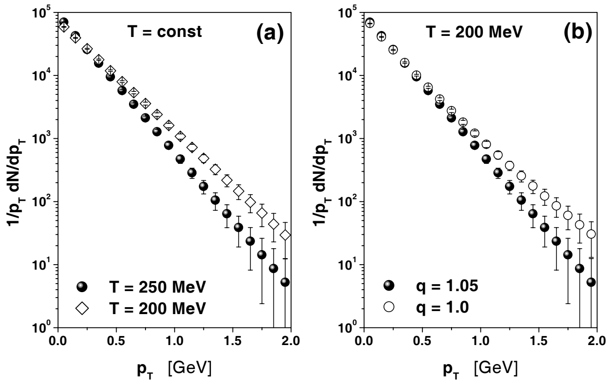

Fig. 6. (a) Normal exponential p_T distributions *i.e.*, $q=1$ for $T=200\,\mathrm{MeV}$ (open symbols) and $T=250\,\mathrm{MeV}$ (black symbols). (b) Typical event from central Pb + Pb at $E_{beam} = 3\ A{\cdot}\mathrm{TeV}$ for $= 200\,\mathrm{MeV}$ for $q=1$ (open symbols) exponential dependence and $q=1.05$ (black symbols). (Reprinted from G. Wilk, Z. Włodarczyk, *Application of nonextensive statistics to particle and nuclear physics*, Physica A **305**, 227 (2002), with kind permission of Elsevier, `http://www.elsevier.com`.)

in fig. 5. On the left-hand panel we can see fits to p_T spectra measured by the UA1 experiment [47] in $p\bar{p}$ at different energies using the Tsallis distribution, eq. (5), with $X/\lambda \to p_T/T$ and with the following values of ($T=T_T$ [GeV], $q=q_T$): (0.134, 1.095), (0.135, 1.105) and (0.14, 1.11) for energies 200, 500 and 900 GeV, respectively (the similar values of the parameter q were obtained in the analysis of transverse momenta in the elementary e^+e^- reaction [23]). These values should be compared with the corresponding values of ($T=T_L; q=q_L$) previously observed for rapidity distributions, which are equal to, respectively: (11.74, 1 .2), (20.39, 1.26) and (30.79, 1.29) at comparable energies, cf. [19].

The right-hand panel of fig. 5 shows an example of a similar behavior observed for nuclear collisions. Such collisions are of special interest as they are the only place where a new state of matter, the Quark Gluon Plasma, can be produced [2] and, because of this, they are intensively investigated using a nonextensive approach (see, for example, [12,21,24,25,28]). As one can see, the best fit is obtained for $q>1$, albeit in this case the value of $(q-1)$, which is the real measure of fluctuations, is noticeably smaller than in the case of the elementary reactions mentioned above. On the other hand, although very small ($|q-1| \sim 0.015$), this deviation leads to a quite substantial relative fluctuations of the temperature existing in the nuclear collisions, namely one gets that $\Delta T/T \simeq 0.12$.

The question then arises: if this is treated seriously, what we are really measuring, what physical observable does it correspond to? It is important to stress that these are fluctuations existing in small parts of the hadronic system in respect to the whole system rather than those of the event-by-event type for which,

$$\Delta T/T = 0.06/\sqrt{N} \to 0$$

for large N. The answer is that the measured fluctuations provide a direct measure of the total heat capacity C of the system [48],

$$\frac{\sigma^2(\beta)}{\langle\beta\rangle^2} = \frac{1}{C} = \omega = q-1, \tag{20}$$

($\beta = \frac{1}{T}$) in terms of $\omega = q-1$. Therefore, single-particle distributions of produced secondaries, if only measured very precisely, can *a priori* provide us information not only on the temperature T of the hadronizing system but also, when investigated using a nonextensive approach, give us information (via value of $q-1$) on its total heat capacity C. In this way one cannot only check whether some (approximate) thermodynamical state is formed in a single collision but also what are its thermodynamical properties —a very important feature, especially in what concerns the existence and type of the possible phase transitions [2].

The next question is: how plausible is such a program? The point is, as discussed above, that one performs fits using T and q in a Tsallis distributions rather than only T in the usual exponential ones. However, the corresponding data on p_T are effectively integrated over the longitudinal phase space (or, at least a part of it) and are averaged over many events. The best thing would be to observe such an effect in single events, then an event-by-event analysis of data would be possible. Figure 6 shows what we can expect. Two scenarios are demonstrated there: (a) T is constant in each event but (because, for example, of different initial conditions) it fluctuates from event to event and (b) T fluctuates in each event around some mean value T_0. We have chosen for comparison a typical event obtained in simulations performed for central Pb + Pb collisions taking place for a beam energy $E_{beam} = 3\ A{\cdot}\mathrm{TeV}$ (expected shortly in the ALICE experiment at LHC). The density of particles in the central region (defined by rapidity window $-1.5 < y < 1.5$) is chosen to be equal to

$\frac{dN}{dy} = 6000$. In case (a) in each event one expects exponential dependence with $T = T_{event}$ and possible departure from it would occur only after averaging over all events. This would reflect fluctuations originating from different initial conditions for each particular collision. This situation is illustrated in fig. 6(a) where p_T distributions for $T = 200$ MeV (open symbols) and $T = 250$ MeV (black symbols) are presented. Such values of T correspond to typical uncertainties in T expected at the LHC accelerator at CERN. Notice that both curves presented here are straight lines. In case (b) one should observe departure from the exponential behavior already on the single-event level and it should be fully given by $q > 1$. This reflects a situation when, due to some intrinsically dynamical reasons, different parts of a given event can have different temperatures, as we have discussed above. In fig. 6(b) the open symbols represent the exponential dependence obtained for $T = 200$ MeV (the same as in fig. 6(a)), the black symbols show the power-like dependence as given by (5) with the same T and with $q = 1.05$ (notice that the corresponding curve bends slightly upward here). In this typical event we have ~ 18000 secondaries, *i.e.*, practically the maximal possible number. Notice that here points with highest p_T already correspond to single particles. As one can see, experimental differentiation between these two scenarios will be very difficult, although not totally impossible. On the other hand, if successful it would be very rewarding[9].

The following remarks are worth to be done at this point.

- We are using rather freely the notion of fluctuating temperature. The question then arises whether it makes sense. Not going into a detailed dispute, we would only like to mention at this point that traces of this idea can already be found in [49,50][10]. In particular it is important when discussing some peculiarities of the phase diagrams, which are important when addressing the question of possible phase transitions between QCD and normal matter [50]. What we want to do is to bring to one's attention the fact that event-by-event analysis allows us (at least *in principle*) to detect fluctuations of temperature taking place *in a given event*. This is more than an indirect measure of fluctuations of T proposed some time ago in [52] or a more direct fluctuations of T *from event to event* discussed in [49].
- As the heat capacity C is proportional to the volume, $C \propto V$, in our case V would be the volume of the interaction (or hadronization), it is expected to grow with volume and, respectively, q is expected to decrease with V. This is indeed the case if one puts together the results for e^+e^-, pp, $p\bar{p}$ and AA collisions for example, those of [23] for e^+e^- collisions, together with those of [6] for $p\bar{p}$ collisions and all the results for heavy-ion collisions, like [24,20,28] and especially [21] where such a trend was found when analyzing heavy-ion events with different centrality (*i.e.*, with different volumes V).
- As the parameter q replaces in some sense the action of many not yet identified dynamical factors, one expects that, with such factors included, q should diminish. This is precisely what has been demonstrated in [21] analyzing transverse momenta of pions produced in RHIC experiments by using a simple-minded Tsallis formula and an accordingly modified Hagedorn [53] approach which already contains in it some dynamics (based on a special bootstrap hypothesis of resonances composed out of resonances and so on). In the second case the values of $q - 1$ found are much smaller, but still remain nonzero indicating therefore the existence of some residual additional dynamics there.
- As demonstrated in [21], using a nonextensive version of the statistical model allows us to well describe data in the domain previously believed to be governed entirely by pure jet physics. Deviations (*i.e.*, dominance of truly hard collisions) start at p_T near 10 GeV and further. It would mean that the so-called mini-jet region can probably also be investigated using a nonextensive approach (what should be, however, checked in more detail in the future).

[9] It is interesting to realize that for the Planckian gas at $T = 186$ MeV, occupying volume of the order of the volume of sulfur nucleus, one gets $C = 34.4$ per degree of freedom, which leads, using eq. (20), to $q = 1.015$ obtained for such system for the p_T-dependence of produced secondaries.

[10] For those interested in the discussion on the problem of internal consistency (or inconsistency) of the notion of fluctuations of temperature in thermodynamics, we refer to [48,49,51].

2.5 The whole phase space

Already presenting results for the longitudinal phase space we encountered multiplicity distributions of produced secondaries. They involve the whole of phase space, both its longitudinal and transverse components. However, as we have already stressed, because $(q_L - 1) \gg (q_T - 1)$, the dominant role of the longitudinal dynamics in establishing the actual number of produced secondaries and its fluctuation from event to event is obvious. We shall now discuss this problem in more detail.

Previous findings could be summarized in the following way: knowing the amount of energy W which is going to be transferred to the produced secondaries (*i.e.*, knowing the inelasticity K of reaction [39]) and the mean number of produced secondaries, $\langle N \rangle$, and respecting the fact that they are essentially distributed in the longitudinal phase space only, one arrives, after using the information theory approach (cf. [5]), at the usual exponential distribution in $E = \langle m_T \rangle \cosh y$. Additional information on the fact that produced secondaries are distributed not according to a Poisson distribution but rather according to an NB distribution characterized by the parameter k is enough to get the q-exponential distribution in E with $q = 1 + 1/k$.

Now, it turns out that the opposite is also true, namely, as we have shown in [22], the fact that N-particles are distributed in energy via the N-particle Tsallis distribution described by the nonextensivity parameter q allows us to show that their number distribution has to be of the NB type with $k = 1/(q-1)$. To illustrate this we first start with the derivation of the Poisson multiplicity distribution and then we compare it with the corresponding derivation of the NB distribution [22].

2.5.1 Poisson multiplicity distribution

This distribution arises in a situation where in some process one has N independently produced secondaries with energies $\{E_{1,\dots,N}\}$, each distributed according to the simple Boltzmann distribution:

$$f(E_i) = \frac{1}{\lambda} \cdot \exp\left(-\frac{E_i}{\lambda}\right) \tag{21}$$

(where $\lambda = \langle E \rangle$). The corresponding joint probability distribution is then given by

$$f(\{E_{1,\dots,N}\}) = \frac{1}{\lambda^N} \cdot \exp\left(-\frac{1}{\lambda}\sum_{i=1}^{N} E_i\right). \tag{22}$$

For independent energies $\{E_{i=1,\dots,N}\}$ the sum $E = \sum_{i=1}^{N} E_i$ is then distributed according to the following gamma distribution:

$$g_N(E) = \frac{1}{\lambda(N-1)!} \cdot \left(\frac{E}{\lambda}\right)^{N-1} \cdot \exp\left(-\frac{E}{\lambda}\right), \tag{23}$$

the distribuant of which is

$$G_N(E) = 1 - \sum_{i=1}^{N-1} \frac{1}{(i-1)!} \cdot \left(\frac{E}{\lambda}\right)^{i-1} \cdot \exp\left(-\frac{E}{\lambda}\right). \tag{24}$$

Equation (23) follows immediately either by using characteristic functions or by sequantially performing integration of the joint distribution (22) and noticing that

$$g_N(E) = g_{N-1}(E)\frac{E}{N-1}. \tag{25}$$

For energies such that

$$\sum_{i=0}^{N} E_i \le E \le \sum_{i=0}^{N+1} E_i \tag{26}$$

the corresponding multiplicity distribution has a Poissonian form (notice that $E/\lambda = \langle N \rangle$):

$$P(N) = G_{N+1}(E) - G_N(E) = \frac{\left(\frac{E}{\lambda}\right)^N}{N!} \cdot \exp(-\alpha E) = \frac{\langle N \rangle^N}{N!} \cdot \exp(-\langle N \rangle). \tag{27}$$

In other words, whenever we have variables $E_{1,\dots,N,N+1,\dots}$ taken from the exponential distribution $f(E_i)$ and whenever these variables satisfy the condition $\sum_{i=0}^{N} E_i \le E \le \sum_{i=0}^{N+1} E_i$, then the corresponding multiplicity N has a Poissonian distribution[11].

[11] Actually, this is the method of generating Poisson distribution in the numerical Monte Carlo codes.

2.5.2 Negative binomial multiplicity distribution

This distribution arises when in some process N independent particles with energies $\{E_{1,\dots,N}\}$, which are distributed according to Tsallis distribution,

$$h(\{E_{1,\dots,N}\}) = C_N\left[1 - (1-q)\frac{\sum_{i=1}^{N} E_i}{\lambda}\right]^{\frac{1}{1-q}+1-N}, \tag{28}$$

with the normalization constant C_N given by

$$C_N = \frac{1}{\lambda^N}\prod_{i=1}^{N}[(i-2)q - (i-3)] = \frac{(q-1)^N}{\lambda^N} \cdot \frac{\Gamma\left(N + \frac{2-q}{q-1}\right)}{\Gamma\left(\frac{2-q}{q-1}\right)}. \tag{29}$$

This means that there are some intrinsic (so far unspecified but summarily characterized by the parameter q) fluctuations present in the system under consideration. In this case we do not know the characteristic function for the Tsallis distribution, however, because we are dealing here only with variables $\{E_{i=1,\dots,N}\}$ occurring in the form of the sum, $E = \sum_{i=1}^{N} E_i$, one can still sequentially perform integrations of the joint probability distribution (28) and, noting that (as before, cf. eq. (25))

$$h_N(E) = h_{N-1}(E)\frac{E}{N-1} = \frac{E^{N-1}}{(N-1)!}h(\{E_{1,\dots,N}\}), \tag{30}$$

we arrive at the formula corresponding to eq. (23), namely

$$h_N(E) = \frac{E^{(N-1)}}{(N-1)!\lambda^N} \times \prod_{i=1}^{N}[(i-1)q - (i-3)]\left[1 - (1-q)\frac{E}{\lambda}\right]^{\frac{1}{1-q}+1-N} \tag{31}$$

with the distribuant given by

$$H_N(E) = 1 - \sum_{j=1}^{N-1} \tilde{H}_i(E), \qquad \text{where}$$
$$\tilde{H}_i(E) = \frac{E^{i-1}}{(j-1)!\lambda^j} \times \prod_{i=1}^{j}[(i-1)q - (i-3)]\left[1 - (1-q)\frac{E}{\lambda}\right]^{\frac{1}{1-q}+1-j}. \tag{32}$$

As before, for energies E satisfying the condition given by eq. (26), the corresponding multiplicity distribution is equal to

$$P(N) = H_{N+1}(E) - H_N(E) \tag{33}$$

and is given by the negative binomial distribution:

$$P(N) = \frac{(q-1)^N}{N!} \cdot \frac{q-1}{2-q} \cdot \frac{\Gamma\left(N+1+\frac{2-q}{q-1}\right)}{\Gamma\left(\frac{2-q}{q-1}\right)} \times \left(\frac{E}{\lambda}\right)^N \left[1-(1-q)\frac{E}{\lambda}\right]^{-N+\frac{1}{1-q}} = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} \cdot \frac{\left(\frac{\langle N\rangle}{k}\right)^N}{\left(1+\frac{\langle N\rangle}{k}\right)^{N+k}}, \quad (34)$$

where the mean multiplicity and variance are, respectively,

$$\langle N\rangle = \frac{E}{\lambda}; \quad \mathrm{Var}(N) = \frac{E}{\lambda}\left[1-(1-q)\frac{E}{\lambda}\right] = \langle N\rangle + \langle N\rangle^2 \cdot (q-1). \quad (35)$$

This distribution is defined by the parameter k:

$$k = \frac{1}{q-1}. \quad (36)$$

Notice that for $q \to 1$ one has $k \to \infty$ and $P(N)$ becomes a Poisson distribution, whereas for $q \to 2$ one has $k \to 1$ and we are obtaining the geometrical distribution[12].

3 Further developments

Let us now proceed a step further in eq. (10) by writing it in the following form:

$$c_p\rho\frac{\partial T}{\partial t} = a(T'-T) + \eta f(u), \quad (37)$$

with a new term, $\eta f(u)$, which presents the effect of a possible viscosity (with viscosity coefficient η) existing in the system. The function $f(u)$ contains terms dependent on the velocity in the form of $\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_l}$. Using as before T' defined by (7) we get an extension of eq. (11):

$$\frac{\partial T}{\partial t} + \left[\frac{1}{\tau} + \xi(t)\right] T = \frac{1}{\tau}T_0 + \eta f(u)\frac{1}{c_p\rho} = \frac{1}{\tau}\left[T_0 + \frac{\eta\tau}{c_p\rho}f(u)\right]. \quad (38)$$

This equation leads to the Langevin equation resulting in fluctuations of the temperature T given in the same form of eq. (12) as before but with

$$\mu = \frac{1}{q-1}\left[T_0 + \frac{\eta\tau}{c_p\rho}f(u)\right] = \frac{T_{eff}}{q-1}. \quad (39)$$

[12] Actually the parameter k in NB can be simply expressed by the correlation coefficient ρ for the two-particle energy correlations, $k = (\rho+1)/\rho$, see [22] for details.

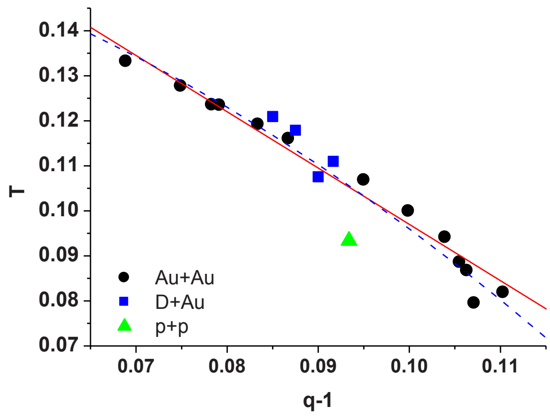

Fig. 7. Dependence of the temperature T (in GeV) on the parameter q for production of negative pions in different reactions. The solid line shows a linear fit to the obtained results: $T = 0.22–1.25(q-1)$ (cf. eq. (41)) and the dashed line shows the corresponding quadratic fit: $T = 0.17–7.5(q-1)^2$ (cf. eq. (42)).

In this way the previous $T_0 = \langle T\rangle$ has now been replaced by a kind of *effective temperature*

$$T_{eff} = T_0 + \frac{\eta\tau}{c_p\rho}f(u) = T_0 + \frac{\eta}{a}f(u). \quad (40)$$

Introducing a kinetic coefficient of conductance $\nu = \eta/\rho$ and denoting $\kappa = c_p/c_V$, where c_V is the specific heat under the constant volume for which $1/c_V = q-1$, we have that

$$T_{eff} = T_0 + \frac{\nu\tau}{\kappa c_V}f(u) = T_0 + (q-1)\frac{\nu\tau}{\kappa}f(u) \quad (41)$$

or, because $\tau D = q-1$, one can write this also as

$$T_{eff} = T_0 + (q-1)^2\frac{\nu}{\kappa D}f(u). \quad (42)$$

In [28] the transverse momentum spectra of pions and protons and antiprotons produced in the interactions of $p+p$, $d+\mathrm{Au}$ and $\mathrm{Au}+\mathrm{Au}$ at $\sqrt{s_{NN}} = 200\,\mathrm{GeV}$ at RHIC-BNL [54] were analyzed using a nonextensive approach. Among other things they found dependences of the nonextensivity parameter q and temperature T on the number of participants, N_p (*i.e.*, the number of nucleons taking part in a given AA collision in the production of secondaries). From them we have obtained a dependence of T on the parameter q which is shown in figs. 7 and 8. In all cases we find that $f(u) < 0$ and that T seems to be linearly dependent on $q-1$.

These results can be compared with old results for e^+e^- annihilation reactions discussed some time ago in terms of q-statistics in [23]. The q-dependence of the temperature parameter T which can be deduced from them is shown in fig. 9. Notice that now the temperature is lower and depends only weakly on q.

Finally, let us discuss results on fluctuations of multiplicity observed in heavy-ion collisions [55]. They exhibit nonmonotonic changes as a function of the number of participants N_p [55]. Actually, also changes of $\langle N\rangle$ show a

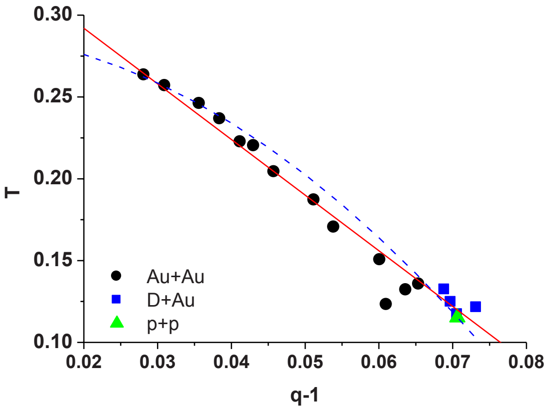

Fig. 8. The same as in fig. 7 but for the produced antiprotons. The linear fit (solid line) is $T = 0.36–3.4(q − 1)$ whereas the quadratic one (dashed line) is $T = 0.29–35(q − 1)^2$.

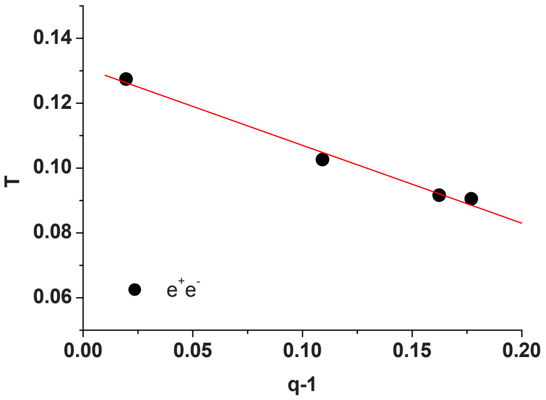

Fig. 9. Dependence of temperature T (in GeV) on the parameter q for the production of pions in e^+e^- annihilation reactions. $T = 0.131–0.24(q − 1)$.

nonlinear increase, though not so spectacular. Acting in the spirit of our analysis here we can expect that

$$\frac{\frac{\mathrm{Var}(N)}{\langle N\rangle} - 1}{\langle N\rangle} = q - 1, \tag{43}$$

but now, with T_{eff} we can show that

$$\langle N\rangle = \frac{W}{T_{eff}}, \tag{44}$$

where T_{eff} is given by eq. (41) and where W is the full accessible energy. We have therefore that

$$\frac{\langle N\rangle - n_0 N_p}{\langle N\rangle} = c(q-1). \tag{45}$$

Here n_0 is the multiplicity in the single nucleon-nucleon collision measured in the region of acceptance, $c = -\frac{\nu\tau}{\kappa}\frac{f(u)}{T_0}$ (notice that c is positive because, as was found from figs. 7 and 8, $f(u) < 0$).

In figs. 10 and 11 we show what can be extracted from PbPb collision data taken by the NA49 experiment [55].

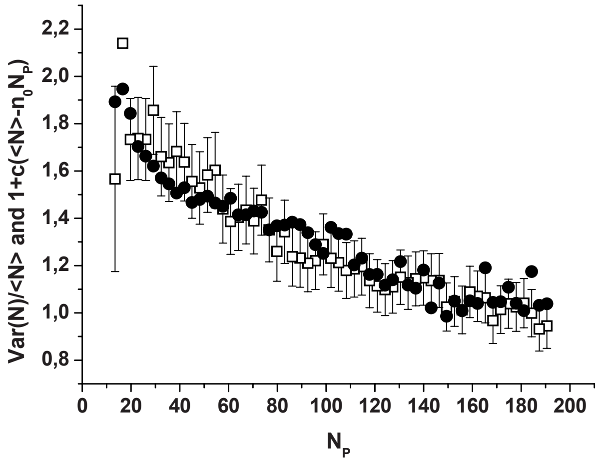

Fig. 10. Comparison of $\mathrm{Var}(N)/\langle N\rangle$ *vs.* N_p (squares) with $1+c(\langle N\rangle − n_0Np)$ *vs.* N_p (circles) (here $n_0 = 0.642$ and $c = 4.1$). Data are for negatively charged particles from PbPb collisions as collected by the NA49 experiment [55].

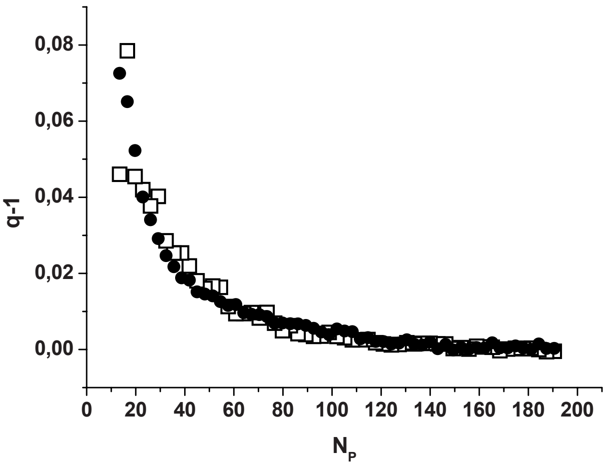

Fig. 11. The same as in fig. 10 but translated to $q − 1$ *vs.* N_p. Squares were obtained from $\mathrm{Var}(N)/\langle N\rangle$ *vs.* N_p and circles from $\langle N\rangle$ *vs.* N_p. As before $n_0 = 0.643$ and $c = 4.1$.

As one can observe the data confirm our expectation that dependences of $\mathrm{Var}(N)/\langle N\rangle$ and $\langle N\rangle$ on the number of participants N_p are, after introducing the concept of T_{eff}, essentially the same. The value of $n_0 = 0.642$ is also sensible, being only a little greater than the multiplicity observed in pp collisions when calculated using the acceptance of the NA49 experiment. Notice also that the value of $c = 4.1$ obtained here for PbPb collisions is not far from the value $1.25/0.22 = 5.7$ obtained for data from RHIC (*i.e.*, for AuAu collisions but at much higher energy) which we have obtained in fig. 7.

We close this part by noticing that this problem is not trivial since none of the known models for multiparticle production processes describes $\mathrm{Var}(N)/\langle N\rangle$ *vs.* N_p observed experimentally [55]. They are described only by some specialized models addressing fluctuations, like the percolation model [56], the model assuming inter-particle correlations caused by the combination of strong and electromagnetic interactions [57] or the transparency, mixing and reflection model [58]. Actually, all those attempts were addressing only $\mathrm{Var}(N)/\langle N\rangle$ *vs.* N_p but not $\langle N\rangle$ *vs.* N_p. From this perspective, results presented above in figs. 10 and 11 confirm the reasonableness of the idea of T_{eff} introduced in this section. If one uses $\mathrm{Var}(N)/\langle N\rangle$ *vs.* N_p to obtain $q-1$ then it turns out that the same value of $q-1$ describes also the dependence of $\langle N\rangle$ on N_p; this can only be using T_{eff} and this is because it depends on $q-1$.

4 Remarks and summary

Let us start with two remarks which are in order here:

i) Results which recall directly to the Tsallis entropy were obtained using the constraint $\sum p_i = 1$ and the formula $\sum_i p_i^q A_i = \langle A\rangle_q$ for the q-expectation values. On the other hand, there exists a formalism, which expresses both the Tsallis entropy and the expectation values using the so-called escort probability distributions [59]: $P_i = p_i^q/\sum_i p_i^q$. However, as was shown in [60], such an approach is different from the normal nonextensive formalism because the Tsallis entropy expressed in terms of the escort probability distributions has some difficulty with the property of concavity. From our limited point of view, it seems that there is no problem in what concerns practical, phenomenological applications of nonextensivity as discussed in the present work. Namely, using P_i one gets distributions of the type $c[1-(1-q)x/l]^{q/(1-q)}$, which is, in fact, *formally identical* with $c[1-(1-Q)x/L]^{1/(1-Q)}$, provided we identify $Q = 1 + (q-1)/q$, $L = l/q$ and $c = (2-Q)/L = 1/l$. The mean value is now $\langle x\rangle = L/(3-2Q) = l/(2-q)$ and $0 < Q < 1.5$ (to be compared with $0.5 < q < 2$). Both distributions are identical and the problem of which of them better describes data is artificial.

ii) One should be aware that there is still an ongoing discussion on the meaning of the temperature in nonextensive systems. However, the small values of the parameter q deduced from data in transverse phase space (where the connection with the thermodynamical approach makes sense, as discussed before) allow us to argue that, to first approximation, T can be regarded as the hadronizing temperature in such a system. One must only remember that in general what we study here is not so much the state of equilibrium but rather some kind of stationary state. For a thorough discussion of the temperature of nonextensive systems, see [61].

With the above reservations in mind, we can summarize that, when looking from the point of view of a statistical approach [1,2], the power law behavior of many distributions observed in elementary and heavy-ion collisions can be traced back to the necessity of using the nonextensive version of a statistical model (here taken in the form proposed by Tsallis [8]).

We interpret this as a sign of some intrinsic fluctuations present in any hadronizing system, which were only recently to be recognized as vital observable when searching for the production of the QGP form of matter [2]. In fact, a number of works [62] have demonstrated the existence in such reactions of event-by-event fluctuations of the average transverse momenta $\langle p\rangle$ per event. The quantities considered were: $\mathrm{Var}(\langle p\rangle)/\langle\langle p\rangle\rangle^2$ and $\langle \Delta p_i \Delta p_j\rangle/\langle\langle p\rangle\rangle^2$. These quantities can be shown [22] to be fully determined by ω as defined by eq. (4), *i.e.*, by fluctuations of the temperature T of the hadronizing system —a vital observable when searching for QGP[13]. In fact, when considering the case of N_{ev} events with N_k particles in the k-th event, one has that

$$\frac{\mathrm{Var}(\langle p\rangle)}{\langle\langle p\rangle\rangle^2} = \frac{\mathrm{Var}(T)}{\langle T\rangle^2} = \omega, \tag{46}$$

where

$$\langle\langle p\rangle\rangle = \frac{1}{N_{ev}}\sum_k^{N_{ev}}\langle p\rangle_k, \quad \text{with} \quad \langle p\rangle_k = \frac{1}{N_k}\sum_i^{N_k} p_i, \tag{47}$$

This is what we have shown in the last part of sect. 3. This is the problem which needs further investigations.

We close with some remarks:

- Although our original investigations presented here were based on the notion of Tsallis entropy (usually with the help of information theory) one must mention that one can also get the Tsallis distribution without resorting to a Tsallis entropy altogether (see, for example, [64]).
- The other way to get a Tsallis distribution from some general thermodynamical considerations was presented in [65]. It is based on allowing a linear dependence of the temperature T on energy, $T = T_0 + (q-1)E$. Here the temperature is not fluctuating. Actually, if one would like to follow this approach and to have the Tsallis distribution with T_{eff} discussed in sect. 3 one should write $T = T_0 + (q-1)\cdot\mathrm{const} + (q-1)E$. Then $T = T_0$ only would result in $\exp(-E/T_0)$, $T = T_0(q-1)E$ would result in the usual Tsallis distribution $\exp_q\,(-E/T_0)$, $T = T_{eff} = T_0 + (q-1)\cdot\mathrm{const}$ would give $\exp(-E/T_{eff})$ and, finally, $T = T_{eff} = T_0 + (q-1)\cdot\mathrm{const} + (q-1)E$ would give $\exp_q(-E/T_{eff})$.
- Notice that for $x \gg \lambda/(q-1)$ the Tsallis distribution becomes a pure power law and loses its dependence on the scale λ: $f(x) \sim [1-(1-q)x/\lambda]^{1/(1-q)} \longrightarrow x^{1/(1-q)}$.
- Instead of using an intrinsic fluctuation one can also obtain a power law distribution by using the notion of self-organized criticality [66] in cascade processes (cf. [16,67]).
- Another interesting possibility, not yet fully explored, is that, as shown in [68], one can formulate a description of the so-called stochastic networks using the nonextensive information theory based on Tsallis

[13] Generally speaking, an analysis of the transverse momenta p_T alone indicates very small fluctuations of T. On the other hand, as reported in [63], the measured fluctuations of the multiplicities of produced secondaries are large (*i.e.*, the multiplicity distributions are substantially broader than Poissonian).

statistics. Using this approach one can then demonstrate [69] that hadron production viewed as the formation of a specific stochastic network can explain in a natural way the power law distributions of transverse mass spectra of pions found in [70].

– In the string models of production of hadrons the natural distribution in p_T is $\exp(-\pi m_T^2/\kappa)$ rather than $\exp(m_T/T)$ (where κ is the string tension) really observed. However, if one allows for the Gaussian fluctuations of the parameter κ (characterized by parameter $\langle\kappa^2\rangle$ which can be connected with the fluctuations in the QCD vacuum) then the first form is transformed into the second one with $T = \sqrt{\langle\kappa^2\rangle/(2\pi)}$ (*i.e.*, in this approach the parameter T characterizes rather the properties of the QCD vacuum than those of hadrons) —see [71].
– Finally, recall that when applied to the hydrodynamical model of multiparticle production the nonextensivity approach converts the usual nonviscous hadronic fluid into the viscous one preserving, however, the usual linear flow equations (albeit now given in a nonextensive form) [72].

Partial support (GW) of the Ministry of Science and Higher Education under contract 1P03B02230 is acknowledged.

References

1. R. Hagedorn, Nuovo Cimento (Suppl.) **3**, 147 (1965); Nuovo Cimento A **52**, 64 (1967); Riv. Nuovo Cimento **6** (1983); C. Geich-Gimbel, Int. J. Mod. Phys. A **4**, 1527 (1989); U. Heinz, J. Phys. G **25**, 263 (1999) 263; F. Becattini, Nucl. Phys. A **702**, 336 (2002); F. Becattini, G. Passaleva, Eur. Phys. J. C **23**, 551 (2002); W. Broniowski, A. Baran, W. Florkowski, Acta Phys. Pol. B **33**, 4325 (2002) 4235.
2. See *QM2008 Proceedings*, J. Phys. G **35**, issue no. 10 (2008); see also, B. Müller, Nucl. Phys. A **774**, 433 (2006); M. Gyulassy, L. McLerran, Nucl. Phys. A **750**, 30 (2005); I. Vitev, Int. J. Mod. Phys. A **20**, 3777 (2005); R.D. Pisarski, Braz. J. Phys. **36**, 122 (2006) and references therein.
3. L. Van Hove, Z. Phys. C **21**, 93 (1985); **27**, 135 (1985).
4. B. Müller, J.L. Nagle, Annu. Rev. Nucl. Part. Sci. **56**, 93 (2006) and references therein.
5. G. Wilk, Z. Włodarczyk, Phys. Rev. D **43**, 794 (1991); O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Acta Phys. Hung. A - Heavy Ion Phys. **25**, 65 (2006).
6. F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Physica A **340**, 467 (2004).
7. C. Arndt, *Information Measures - Information and its Description in Science and Engineering* (Springer, 2004); T.I.J. Taneja, *Generalized Information Measures and Their Applications*, on-line book, `http://www.mtm.ufsc.br/taneja/book/book.html`.
8. C. Tsallis, J. Stat. Phys. **52**, 479 (1988); Braz. J. Phys. **29**, 1 (1999); Physica A **340**, 1 (2004); **344**, 718 (2004) and references therein. See also S. Abe, Y. Okamoto (Editors), *Nonextensive Statistical Mechanics and its Applications*, Lect. Notes Phys., Vol. **560** (Springer, 2000); M. Gell-Mann, C. Tsallis (Editors), *Nonextensive Entropy - Interdisciplinary Applications* (a volume in the Santa Fe Institute Studies in the Science of Complexity) (Oxford University Press, 2004). For an updated bibliography on this subject, see `http://tsallis.cat.cbpf.br/biblio.htm`; C. Tsallis, M. Gell-Mann, Y. Sato, Europhys. News **36**, 186 (2005).
9. T. Kodama, H.-T. Elze, C.E. Augiar, T. Koide, Europhys. Lett. **70**, 439 (2005); T. Kodama, J. Phys. G **31**, S1051 (2005).
10. T. Laštovička, Eur. Phys. J. C **24**, 529 (2002).
11. G. Wilk, Z. Włodarczyk, Phys. Rev. Lett. **84**, 2770 (2000).
12. G. Wilk, Z. Włodarczyk, Chaos, Solitons Fractals **13**, 581 (2001).
13. F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Nuovo Cimento C **24**, 725 (2001).
14. C. Beck, E.G.D. Cohen, Physica A **322**, 267 (2003); F. Sattin, Eur. Phys. J. B **49**, 219 (2006).
15. O.V. Utyuzh, G. Wilk, Z. Włodarczyk, J. Phys. G **26**, L39 (2000).
16. G. Wilk, Z. Włodarczyk, Physica A **305**, 227 (2002).
17. M. Rybczyński, Z. Włodarczyk, G. Wilk, Nucl. Phys. B Proc. Suppl. **122**, 325 (2003).
18. T. Osada, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Eur. Phys. J. B **50**, 7 (2006).
19. F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Phys. Rev. D **67**, 114002 (2003).
20. F.S. Navarra, O.V. Utyuzh, G. Wilk, Z. Włodarczyk, Physica A **344**, 568 (2004).
21. M. Biyajima, M. Kaneyama, T. Mizoguchi, G. Wilk, Eur. Phys. J. C **40**, 243 (2005); M. Biyajima, T. Mizoguchi, N. Nakajima, N. Suzuki, G. Wilk, Eur. Phys. J. C **48**, 593 (2006).
22. G. Wilk, Z. Włodarczyk, Physica A **376**, 279 (2007).
23. I. Bediaga, E.M. Curado, J.M. de Miranda, Physica A **286**, 156 (2000).
24. W.M. Alberico, A. Lavagno, P. Quarati, Eur. Phys. J. C **12**, 499 (2000).
25. T. Wibig, I. Kurp, J. High Energy Phys. **12**, 039 (2003).
26. A. Lavagno, Physica A **305**, 238 (2002); W.M. Alberico, P. Czerski, A. Lavagno, M. Nardi, V. Somá, Physica A **387**, 467 (2008).
27. C.E. Aguiar, T. Kodama, Physica A **320**, 371 (2003).
28. B. De, S. Bhattacharyya, G. Sau, S.K. Biswas, Int. J. Mod. Phys. E **16**, 1687 (2007).
29. T. Sherman, J. Rafelski, Lect. Notes Phys. **633**, 377 (2004).
30. T.S. Biró, G. Purcsel, Phys. Rev. Lett. **95**, 162302 (2005); Phys. Lett. A **372**, 1174 (2008). See also T.S. Biró, EPL **84**, 56003 (2008).
31. T.S. Biró, G. Kaniadakis, Eur. Phys. J. B **50**, 3 (2006) and references therein.
32. C. Beck, Physica A **286**, 164 (2000).
33. G. Wilk, Z. Włodarczyk, Nucl. Phys. B Proc. Suppl. **75A**, 191 (1999).
34. G. Wilk, Z. Włodarczyk, Phys. Rev. D **50**, 2318 (1994).
35. H. Heiselberg *et al.*, Phys. Rev. Lett. **67**, 2946 (1991); B. Blättel *et al.*, Phys. Rev. D **47**, 2761 (1993); L. Frankfurt, V. Guzey, M. Strikman, J. Phys. G **27**, R23 (2001).
36. L. Frankfurt, M. Strikman, D. Treleani, C. Weiss, Phys. Rev. Lett. **101**, 202003 (2008).

37. L.D. Landau, I.M. Lifschitz, *Course of Theoretical Physics: Hydrodynamics* (Pergamon Press, New York, 1958); *Course of Theoretical Physics: Mechanics of Continous Media* (Pergamon Press, Oxford, 1981).
38. T.S. Biró, A. Jakovác, Phys. Rev. Lett. **94**, 132302 (2005).
39. Y.-A. Chao, Nucl. Phys. B **40**, 475 (1972); Y.M. Shabelski, R.M. Weiner, G. Wilk, Z. Włodarczyk, J. Phys. G **18**, 1281 (1992); F.O. Durães, F.S. Navarra, G. Wilk, Braz. J. Phys. **35**, 3 (2005).
40. C. De Marzo *et al.*, Phys. Rev. D **26**, 1019 (1982); **29**, 2476 (1984); R. Baltrusaitis *et al.*, Phys. Rev. Lett. **52**, 1380 (1993); F. Abe *et al.*, Phys. Rev. D **41**, 2330 (1990).
41. ALEPH Collaboration (R. Barate *et al.*), Phys. Rep. **294**, 1 (1998).
42. T.T. Chou, C.N. Yang, Phys. Rev. Lett. **54**, 510 (1985); Phys. Rev. D **32**, 1692 (1985).
43. P. Carruthers, C.S. Shih, Int. J. Mod. Phys. A **2**, 1447 (1986).
44. NA49 Collaboration (S.V. Afanasjev *et al.*), Phys. Rev. C **66**, 054902 (2002).
45. PHOBOS Collaboration (B.B. Beck *et al.*), Phys. Rev. Lett. **91**, 052303 (2003).
46. UA1 Collaboration (C. Albajar *et al.*), Nucl. Phys. B **335**, 261 (1990).
47. NA35 Collaboration (T. Alber *et al.*), Eur. Phys. J. C **2**, 643 (1998).
48. L.D. Landau, I.M. Lifschitz, *Course of Theoretical Physics: Statistical Physics* (Pergmon Press, New York, 1958).
49. L. Stodolsky, Phys. Rev. Lett. **75**, 1044 (1995); S. Mrówczyński, Phys. Lett. B **430**, 9 (1998); E.V. Shuryak, Phys. Lett. B **423**, 9 (1998).
50. M. Stephanov, K. Rajagopal, E. Shuryak, Phys. Rev. Lett. **81**, 4816 (1998); Phys. Rev. D **60**, 114028 (1999); S. Mrówczyński, Phys. Rev. C **57**, 1518 (1998).
51. T.C.P. Chui, D.R. Swanson, M.J. Adriaans, J.A. Nissen, J.A. Lipa, Phys. Rev. Lett. **69**, 3005 (1992); C. Kittel, Phys. Today **5**, 93 (1988); B.B. Mandelbrot, Phys. Today **42**, 71 (1989); H.B. Prosper, Am. J. Phys. **61**, 54 (1993); G.D.J. Phillies, Am. J. Phys. **52**, 629 (1984).
52. K. Kadaja, M. Martinis, Z. Phys. C **56**, 437 (1992).
53. R. Hagedorn, Nuovo Cimento Suppl. **3**, 147 (1965); Nuovo Cimento A **52**, 64 (1967); CERN Report 71-12 (1971).
54. PHENIX Collaboration (S.S. Adler *et al.*), Phys. Rev. C **69**, 034909 (2004); STAR Collaboration (J. Adams *et al.*), Phys. Lett. B **616**, 8 (2005); **637**, 161 (2006).
55. NA49 Collaboration (C. Alt *et al.*), Phys. Rev. C **75**, 064904 (2007).
56. E.G. Ferreiro, F. del Moral, C. Pajares, Phys. Rev. C **69**, 034901 (2004).
57. M. Rybczyński, Z. Włodarczyk, J. Phys. Conf. Ser. **5**, 238 (2005).
58. M. Gaździcki, M. Gorenstein, Phys. Lett. B **640**, 155 (2006).
59. C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A **261**, 534 (1998).
60. S. Abe, Phys. Lett. A **275**, 250 (2000).
61. S. Abe, Physica A **368**, 430 (2006).
62. W. Broniowski, B. Hiller, W. Florkowski, P. Bożek, Phys. Lett. B **635**, 290 (2006); F. Jinghua *et al.*, Phys. Rev. C **72**, 017901 (2005); STAR Collaboration (J. Adams *et al.*), Phys. Rev. C **72**, 044902 (2005); PHENIX Collaboration (K. Adcox *et al.*), Phys. Rev. C **66**, 024901 (2002).
63. NA49 Collaboration (M. Rybczyński *et al.*), J. Phys. Conf. Ser. **5**, 74 (2005).
64. G. Wilk, Z. Włodarczyk, AIP Conf. Proc. **965**, 76 (2007).
65. M.P. Almeida, Physica A **300**, 424 (2001); **325**, 426 (2003).
66. Fu Jinghua, Meng Ta-chung, R. Rittel, K. Tabelow, Phys. Rev. Lett. **86**, 1961 (2001).
67. M. Rybczyński, Z. Włodarczyk, G. Wilk, Nucl. Phys. B Proc. Suppl. **97**, 81 (2001).
68. G. Wilk, Z. Włodarczyk, Acta Phys. Pol. B **35**, 871 (2004).
69. G. Wilk, Z. Włodarczyk, Acta Phys. Pol. B **35**, 2141 (2004).
70. M. Gaździcki, M.I. Gorenstein, Phys. Lett. B **517**, 250 (2001).
71. A. Białas, Phys. Lett. B **466**, 301 (1999).
72. T. Osada, G. Wilk, Phys. Rev. C **77**, 044903; Prog. Theor. Phys. Suppl. **174**, 168 (2008); see also *Dissipative or just Nonextensive hydrodynamics? Nonextensive/Dissipative correspondence*, arXiv:0805.2253 [nucl-phys], to be published in Indian J. Phys.