Total Hadronic Cross Section in e^+e^- Annihilation at the Four-Loop Level of Perturbative QCD

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The ratio $R(S)=\sigma_{\rm tot}(e^+e^-\to \text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$ is calculated to the four-loop level in perturbative QCD. The result differs from a previous calculation and is much smaller. In addition, a new result for the four-loop β function in QED is obtained. This result also disagrees with a previous calculation.

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The annihilation of e^+e^- into hadrons is one of the most important processes for testing the theory of strong interactions—QCD (for a recent review see Ref. 1). This process provides a fundamental QCD test, providing evidence for the existence of color[2] (see also Ref. 3). Moreover, the comparison of the theoretical and experimental results allows one to extract the fundamental parameters of the theory, such as the strong coupling $\alpha_s(Q^2)$ and the QCD scale parameter $\Lambda_{\rm QCD}$.

The main observable of the above process is the total cross section, known at present from experiments at e^+e^- colliders.[4] On the other hand, the most convenient characteristic of the process is the ratio

$$R(s)=\sigma_{\rm tot}(e^+e^-\to \text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)\,.$$

This quantity can be calculated theoretically in perturbative QCD. Indeed, a well-known dispersion relation allows us to connect $R(s)$ to the hadronic vacuum-polarization function $\Pi(Q^2)$ (see, for example, Ref. 3):

$$-\frac{3}{4}Q^2\frac{d}{dQ^2}\Pi(Q^2)=Q^2\int_0^\infty \frac{R(s)}{(s+Q^2)^2}ds\,, \qquad (1)$$

where the left-hand side can be calculated by perturbation theory in deep Euclidean space. After transformation of the results into the physical region[5] we obtain

$$R(s)=R_0+(\bar\alpha_s/\pi)R_1+(\bar\alpha_s/\pi)^2R_2 + (\bar\alpha_s/\pi)^3\left(R_3-3\sum_f Q_f^2\pi^2\beta_0^2/3\right)+O(\bar\alpha_s^4)\,, \qquad (2)$$

where $\bar\alpha_s$ is the running coupling constant. β_0 is the first coefficient of the QCD β function. The additional scheme-independent correction, proportional to π^2, is due to the procedure of analytical continuation of the results of the perturbative calculation of $d\Pi(Q^2)/dQ^2$ to the physical region.[5]

The lowest-order coefficient R_0 is known from the parton model.[3] The leading QCD correction R_1 was computed a long time ago in the zero-quark-mass limit.[3] The $O(\alpha_s)$ correction for massive quarks has also been calculated.[6]

The dimensional-regularization method[7] and the corresponding renormalization procedure presented by 't Hooft,[8] the so-called "minimal-subtraction (MS) scheme," will be used. Investigation of the renormalization properties within the above scheme,[9] in combination with the recent progress in calculational methods and algorithms for relevant types of Feynman diagrams,[10] the infrared-rearrangement technique,[11-13] and the computer implementation of the above algorithms[14,15] [the algebraic programming systems used are REDUCE (Ref. 16) and SCHOONSCHIP (Ref. 17)], allow one to perform the renormalization-group calculations up to and including the four-loop level. Using the methods enumerated above, the three-loop quantity R_2 was computed analytically in Ref. 18 and numerically in Ref. 19.

Recently, the results of a calculation of $R(s)$ to $O(\alpha_s^3)$ was given in Ref. 20. The numerical value of the coefficient R_3, which is given in Ref. 20, is very large. This casts doubt on the feasibility of obtaining reliable estimates for QCD corrections to $R(s)$ via perturbation theory. (The same effect was observed in perturbative calculations of the coefficient functions of quark condensates in QCD sum rules.[21]) Inclusion of the $O(\alpha_s^3)$ correction changes the value of α_s by about 10%.[1] The corresponding value of $\Lambda_{\overline{\rm MS}}$ decreases drastically (by a factor of 2). However, further consideration shows that the result of Ref. 20 for the $O(\alpha_s^3)$ correction is not correct.

In this work we present the results of an independent reevaluation of the four-loop correction to $R(s)$, by using the methods, algorithms, and programs of Refs. 7–17.

It is known that the vacuum-polarization function is renormalized additively:

$$\Pi\left(\frac{Q^2}{\mu^2},\alpha_s\right)=\Pi^B\left(\frac{Q^2}{\mu^2},\alpha_s^B\right)+Z\,, \qquad (3)$$

where the bare coupling α_s^B is related to the renormalized one by the relation

$$\alpha_s^B = \alpha_s \left[1 - \frac{\alpha_s}{4\pi} \frac{\beta_0}{\varepsilon} + \left(\frac{\alpha_s}{4\pi} \right)^2 \left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon} \right) \right]. \tag{4}$$

Using Eqs. (1)-(4) and the Bogoliubov-Parasiuk theorem which ensures the cancellation of divergences, it is not difficult to express the coefficients R_i in terms of the coefficients of the renormalization constant Z,

$$Z = \sum_{-l \le k < 0} \alpha_s^{l-1} Z_{lk} \varepsilon^k , \tag{5}$$

and the perturbative coefficients of the vacuum-polarization function,

$$\Pi(Q^2) = \sum_{-l \le k,\, l > 0} \alpha_s^{l-1} (\mu^2/Q^2)^{l\varepsilon} \Pi_{lk} \varepsilon^k . \tag{6}$$

After some simple algebraic manipulations, we obtain our main expression for $R(s)$:

$$R(s) = -\tfrac{3}{4} \Big[Z_{1,-1} + 2Z_{2,-1}(\bar{\alpha}_s/4\pi) + (3Z_{3,-1} - \beta_0 \Pi_{2,0})(\bar{\alpha}_s/4\pi)^2 + \Big(4Z_{4,-1} - 2\beta_0\Pi_{3,0} - \beta_1 \Pi_{2,0} + 2\beta_0^2 \Pi_{2,1} - 3\sum_f Q_f^2 \pi^2 \beta_0^2/3 \Big) (\bar{\alpha}_s/4\pi)^3 + O(\alpha_s^4) \Big] , \tag{7}$$

where the first index denotes the number of loops of the corresponding Feynman diagrams. The expression (7) shows that in order to calculate the l-loop contribution to R, one should calculate the l-loop counterterm Z to the bare quantity Π^B, and the $(l-1)$-loop approximation to Π^B.

The essence of the various versions of the infrared-rearrangement procedure,[11,13] which is used in the present work is the following: The problem of calculating the counterterm of an arbitrary l-loop diagram with an arbitrary number of masses and external momenta within the MS scheme can be reduced, through infrared rearrangement, to the problem of calculating some $(l-1)$-loop massless integrals to $O(\varepsilon^0)$ with only one external momentum.

Using the infrared-rearrangement procedure,[13] we have reduced our calculation to the evaluation of the three-loop massless diagrams with one external momentum, up to and including $O(\varepsilon^0)$ terms in the corresponding Laurent expansion in ε. Analytical calculation of relevant diagrams can be performed by using the program MINCER.[15]

The renormalization constant Z can be found from the following relation:[11]

$$Z = 1 - \hat{K}\hat{R}'\Pi^B(Q^2/\mu^2, \alpha_s^B) , \tag{8}$$

where the operator $\hat{K}$ picks out all singular terms from the Laurent series in ε. $\hat{R}'$ is the ordinary Bogoliubov-Parasiuk $\hat{R}$ operation without the last subtraction,[11]

$$\hat{R} = (1 - \hat{K})\hat{R}' . \tag{9}$$

The total number of topologically distinct four-loop diagrams contributing to $Z_{4,i}$ is 98. However, after application of the infrared-rearrangement procedure, which involves differentiating twice with respect to the external momentum of the diagram,[13] the number of four-loop diagrams which need to be calculated increases to approximately 250. Furthermore, there are one- and two-loop diagrams approximately 600, which need to be calculated in order to subtract subdivergences (evaluate $\hat{R}'$) for all four-loop diagrams.

All calculations have been done by using the program MINCER (Ref. 15), and the evaluation of one- and two-loop counterterms has been performed by the program LOOPS.[14] The above programs are written for the algebraic programming systems SCHOONSCHIP (Ref. 17) and REDUCE,[16] respectively. The total CPU time on three 0.8-Mflop EC-1037 computers was over 700 h.

We obtain the following analytical result for $R(s)$ in QCD in the modified minimal-subtraction ($\overline{\text{MS}}$) scheme at the four-loop level:

$$\begin{aligned} R^{\overline{\text{MS}}}(s) = 3\sum_f Q_f^2 \Big(1 &+ (\bar{\alpha}_s/4\pi)(3C_F) + (\bar{\alpha}_s/4\pi)^2 \{ C_F^2(-\tfrac{3}{2}) + C_F C_A[\tfrac{123}{2} - 44\zeta(3)] + C_F T N_f[-22 + 16\zeta(3)] \} \\ &+ (\bar{\alpha}_s/4\pi)^3 \{ C_F^3(-\tfrac{69}{2} + C_F^2 C_A[-127 - 572\zeta(3) + 880\zeta(5)] + C_F C_A^2[+\tfrac{90445}{54} - \tfrac{10948}{9}\zeta(3) - \tfrac{440}{3}\zeta(5)] \\ &+ C_F^2 T N_f[-29 + 304\zeta(3) - 320\zeta(5)] + C_F C_A T N_f[-\tfrac{31040}{27} + \tfrac{7168}{9}\zeta(3) + \tfrac{160}{3}\zeta(5)] \\ &+ C_F T^2 N_f^2[+\tfrac{4832}{27} - \tfrac{1216}{9}\zeta(3)] - \tfrac{4}{3}\pi^2(\tfrac{11}{3}C_A - \tfrac{4}{3}TN_f)^2 \} + O(\alpha_s^4) \Big) \\ &+ (\bar{\alpha}_s/4\pi)^3 \Big(\sum_f Q_f \Big)^2 (d_{abc}/4)^2[+\tfrac{176}{3} - 128\zeta(3)] , \end{aligned} \tag{10}$$

where Q is the quark charge. For the $SU_c(3)$ gauge group, $C_F = \frac{4}{3}$, $C_A = 3$, $T = \frac{1}{2}$, and $(d_{abc}/4)^2 = \frac{5}{6}$. f enumerates quark flavors; their total number is N_f. The momentum integrations are performed using the $\overline{\text{MS}}$ prescription, which amounts to formally setting $\gamma=\zeta(2)=\ln 4\pi=0$. Substituting the values for group invariants we get

$$\begin{aligned} R^{\overline{\mathrm{MS}}}(s) = 3\sum_f Q_f^2 \Big(1 + \bar{\alpha}_s/\pi + (\bar{\alpha}_s/\pi)^2\{+\tfrac{365}{24} - 11\zeta(3) - N_f[\tfrac{11}{12} - \tfrac{2}{3}\zeta(3)]\} \\ + (\bar{\alpha}_s/\pi)^3\{+\tfrac{87029}{288} - \tfrac{1103}{4}\zeta(3) + \tfrac{275}{6}\zeta(5) + N_f[-\tfrac{7847}{216} + \tfrac{262}{9}\zeta(3) - \tfrac{25}{9}\zeta(5)] \\ + N_f^2[+\tfrac{151}{162} - \tfrac{19}{27}\zeta(3)] - \pi^2/48(11 - \tfrac{2}{3}N_f)^2\} + O(\alpha_s^4)\Big) \\ + \left(\sum_f Q_f\right)^2 (\bar{\alpha}_s/\pi)^3[\tfrac{55}{72} - \tfrac{5}{3}\zeta(3)] + O(\alpha_s^4). \end{aligned} \tag{11}$$

Taking into account the values for relevant Riemann ζ functions $\zeta(3)=1.2020569$ and $\zeta(5)=1.0369278$ we obtain the numerical form

$$\begin{aligned} R^{\overline{\mathrm{MS}}}(s) = 3\sum_f Q_f^2[1 + \bar{\alpha}_s/\pi + (\bar{\alpha}_s/\pi)^2(1.986 - 0.115N_f) + (\bar{\alpha}_s/\pi)^3(-6.637 - 1.200N_f - 0.005N_f^2) + O(\alpha_s^4)] \\ - \left(\sum_f Q_f\right)^2 (\bar{\alpha}_s/\pi)^3 1.240 + O(\alpha_s^4). \end{aligned} \tag{12}$$

The last term, which is proportional to $(\sum_f Q_f)^2$ comes from the "light-by-light" type diagrams. Terms of such type appear only at the four-loop level and in higher order.

As seen, the results (11) and (12) are smaller by an order of magnitude and have the opposite sign from the results of Ref. 20. The terms of $O(N)$ and $O(N^2)$ coincide with the corresponding terms of the previous results.[20]

There are two sources of error in the program[22] which was used in the previous calculation.[20] The first is an incorrect trace calculation, and the second is due to an error in the subroutine which calculates the expansions of the Laurent series in ε, of the one-loop integrals. Unfortunately, both errors affect the terms of $O(\varepsilon^0)$ only and could not be eliminated by the tests of cancellations of poles, including those proportional to logarithmic terms, which were done.

Including u, d, c, s, and b quarks ($N_f=5$) we obtain the following result for $R^{\overline{\mathrm{MS}}}(s)$:

$$R^{\overline{\mathrm{MS}}}(s) = \frac{11}{3}\left[1 + \frac{\bar{\alpha}_s}{\pi} + 1.409\left(\frac{\bar{\alpha}_s}{\pi}\right)^2 - 12.805\left(\frac{\bar{\alpha}_s}{\pi}\right)^3 + O(\bar{\alpha}_s^4)\right]. \tag{13}$$

Finally, we obtain the result for α_s,

$$\alpha_s(34\ \mathrm{GeV}) = 0.148(16), \tag{14}$$

and for $\Lambda_{\overline{\mathrm{MS}}}$,

$$\Lambda_{\overline{\mathrm{MS}}} = 223^{+160}_{-109}\ \mathrm{MeV}. \tag{15}$$

As an intermediate result of our calculation we also present the QED β function at the four-loop level in the $\overline{\mathrm{MS}}$ scheme:

$$\beta_{\mathrm{QED}}(\alpha) = \frac{4}{3}N\left(\frac{\alpha}{4\pi}\right)^2 + 4N\left(\frac{\alpha}{4\pi}\right)^3 - N\left(2 + \frac{44}{9}N\right)\left(\frac{\alpha}{4\pi}\right)^4 - N\left\{46 + \left[\frac{760}{27} + \frac{832}{9}\zeta(3)\right]N + \frac{1232}{243}N^2\right\}\left(\frac{\alpha}{4\pi}\right)^5, \tag{16}$$

where $\alpha=e^2/4\pi$ and e is the electron electric charge. N is the number of fermions. The total number of topologically distinct four-loop diagrams which contribute to β_{QED} is 58. The result also differs from the earlier result for the four-loop β_{QED}.[23]

The reevaluation of β_{QED} at the four-loop level has also been performed independently by the authors of the original work[23] (A.L.K. and S.A.L.) at the Joint Institute for Nuclear Research (JINR, Dubna) computer center. The results of both calculations agree. The details of the calculations and the results will be given in a joint publication.[24]

In conclusion, we note that we have obtained the four-loop correction to

$$R(s) = \sigma_{\mathrm{tot}}(e^+e^- \to \mathrm{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-).$$

Our result is quite small and clearly differs from the incorrect result of Ref. 20. Therefore, all of the phenomenological applications which used the result of Ref. 20 should be revised.

More details of our calculations, graph-by-graph results, and some phenomenological applications will be given elsewhere.

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