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# ASYMPTOTIC FREEDOM IN PARTON LANGUAGE

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A novel derivation of the $Q^2$ dependence of quark and gluon densities (of given helicity) as predicted by quantum chromodynamics is presented. The main body of predictions of the theory for deep-inleastic scattering on either unpolarized or polarized targets is re-obtained by a method which only makes use of the simplest tree diagrams and is entirely phrased in parton language with no reference to the conventional operator formalism.

## 1. Introduction

The quark parton model [1] provides us with a very useful and simple description of the physics of deep inelastic phenomena [2]. The theoretical framework which justifies the parton model is given by the asymptotically free gauge theory of strong interactions based on the color degrees of freedom [3] (quantum chromodynamics, QCD). Although scaling is predicted to be broken by logarithms (a fact which appears to be well consistent with present experiments), the deviations from scaling can be and have been computed for deep inelastic structure functions for either unpolarized [4,5] or polarized targets [6,7]. In the leading logarithmic approximation, the results can again be phrased in the parton language by assigning a well determined $Q^2$ dependence to the parton densities. In spite of the relative simplicity of the final results, their derivation, although theoretically rigorous, is somewhat abstract and formal, being formulated in the language of renormalization group equations for the coefficient functions of the local operators which appear in the light cone expansion for the product of two currents.

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In this paper we show that an alternative derivation of all results of current interest for the $Q^2$ behaviour of deep inelastic structure functions is possible. In this approach all stages of the calculation refer to parton concepts and offer a very illuminating physical interpretation of the scaling violations. In our opinion the present approach, although less general, is remarkably simpler than the usual one since all relevant results can be derived in a direct way from the basic vertices of QCD, with no loop calculations being involved (the only exception is the lowest order expression for the running coupling constant which we do not rederive).

This method can be described as an appropriate generalization of the equivalent photon approximation in quantum electrodynamics [8]. A preliminary and less complete version of this paper has been presented by one of us at the 1976 Flaine meeting [9].

The present paper is organized as follows. In sect. 2 the basic formulae for deep inelastic processes are recalled and notations and definitions are specified. In sect. 3 the integro-differential equations that describe the $Q^2$ dependence of parton densities are introduced and their physical interpretation is discussed at length. In sect. 4 the calculation of logarithmic exponents for the spin averaged structure functions is described, while sect. 5 is devoted to the calculation of the other logarithmic exponents which are relevant to spin dependent structure functions. Sect. 6 contains some concluding remarks.

## 2. Basic formulae and definitions

In this section we recall some basic notions and notations concerning deep inelastic scattering on a polarized target which are of relevance for the following. For simplicity in all of this paper we refer to the case of electroproduction, the extension to neutrino scattering being simple and well known.

Let $p$ and $q$ be the four-momenta of the nucleon (of mass $M$) and of the virtual photon respectively. As usual we define:

$$M\nu = (pq)\,, \qquad -q^2 = Q^2 > 0\,, \qquad 2M\nu x = Q^2\,. \tag{1}$$

The structure functions of electroproduction are defined as [1,6]

$$\begin{aligned}&4\pi^2 \frac{E}{M}\int \mathrm{d}y e^{iqy} \langle p, s|[J_\mu(\tfrac{1}{2}y), J_\nu(-\tfrac{1}{2}y)]|p,s\rangle \\ &= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x, Q^2) + \left(p_\mu - \frac{M\nu}{q^2}q_\mu\right)\left(p_\nu - \frac{M\nu}{q^2}q_\nu\right)\frac{W_2(x, Q^2)}{M^2} \\ &-i\epsilon_{\mu\nu\lambda\sigma}\, q^\lambda\left[s^\sigma \frac{V_1(x, Q^2)}{M} + (M\nu s^\sigma - (qs)p^\sigma)\frac{V_2(x, Q^2)}{M^3}\right],\end{aligned} \tag{2}$$

where $s^\sigma$ is the polarization four-vector of the nucleon. The number of independent structure functions is fixed by the number of non-vanishing independent helicity amplitudes for forward virtual Compton scattering in the case of parity and time reversal invariance.

Standard arguments based on free quark light cone expansion [10] or the parton model [1] show that

$$MW_1 \equiv F_1\,, \qquad \nu W_2 \equiv F_2\,, \tag{3}$$

$$\nu V_1 \equiv G_1\,, \qquad \nu^2 V_2 \equiv MG_2\,, \tag{4}$$

are finite and independent of $Q^2$ in the scaling approximation (that is when logarithmic effects from strong interactions are neglected). In this same approximation it is also true that

$$2xF_1 = F_2 \tag{5}$$

as a consequence of the vanishing of the longitudinal cross section. Moreover, the structure functions $F_1$, $F_2$, and $G_1$ can be simply expressed in terms of parton quark densities. Let $q^i(x)$ be the number density of quarks of type $i$ (summed over colors) inside a proton target with fraction $x$ of the proton longitudinal momentum in the $p_\infty$ frame. Similarly we denote, by $q^i_+(x)$ and $q^i_-(x)$ the densities of parton quarks with positive and negative helicities in a proton of positive helicity. One has, of course,

$$q^i_+(x) + q^i_-(x) = q^i(x)\,. \tag{6}$$

Defines $e_i$ as the charge of the $i$th quark in units of the proton charge, we have [1]

$$2F_1 = F_2/x = \sum_i e_i^2\,[q^i(x) + \bar{q}^i(x)]\,, \tag{7}$$

$$2G_1 = \sum_i e_i^2\,[q^i_+(x) - q^i_-(x) + \bar{q}^i_+(x) - \bar{q}^i_-(x)]\,. \tag{8}$$

The structure function $G_2$ does not have an equally simple expression in terms of quark densities of given helicity. This is related to the fact that $G_2$ would be zero for a point-like proton (namely a proton that is itself a parton), as one can easily see from the Born diagram of virtual Compton scattering. In light cone language one finds that $G_2$ would be zero if matrix elements of local operators were extracted from free quark field theory. This is certainly not admissible in general. However our experience (for example in the case of 'sea' densities) strongly suggests that $G_2$ will turn out to be a small effect.

We recall that $G_2$ contributes [10], quite in general, corrections of order $1/Q^2$ to the asymmetry in the case of longitudinally (with respect to the beam) polarized protons. In this case the dominant contribution of order one is determined by $G_1$. For transversely polarized protons, $G_1$ and $G_2$ contribute [10] terms of the same order to the asymmetry which however is itself of order $1/Q$ in this case.

In this paper we shall restrict ourselves to a discussion of scale breaking effects concerning $F_1$, $F_2$ and $G_1$ which have a simplest description in parton language. In fact in the following sections we shall show that the $Q^2$ dependence of the densities $q^i_+ + q^i_-$ and $q^i_+ - q^i_-$ for all types of quarks and antiquarks can be directly evaluated in QCD on the basis of simple physical arguments. From the previous discussion we see that this knowledge, although not complete is however sufficient to reproduce all the results of the theory which are of current interest.

## 3. The master equations

In this section we shall introduce a set of integro-differential equations for the $Q^2$ dependence of quark and gluon densities which are the basis for the present approach. These equations were originally obtained [11] by Mellin transform techniques starting from the renormalization group equations. Hereafter briefly recalling their derivation we discuss their physical contents [12]. We shall deal first with the unpolarized case when only the densities

$$q^i(x) = q^i_+(x) + q^i_-(x)$$

are relevant. The further extension to the general case will be done in sect. 5.

We start by considering the simplest case of only one flavour of quarks. We denote by $q^{\mathrm{NS}}$ the "net" number of quarks in the proton as seen by the current in the $p_\infty$ frame, that is the number of quarks minus antiquarks:

$$q^{\mathrm{NS}}(x, t) = q(x, t) - \bar{q}(x, t) \,. \tag{9}$$

The variable $t$ is defined as

$$t \equiv \ln Q^2/Q_0^2 \,, \tag{10}$$

with $Q_0^2$ a suitable normalization point. The label NS stands for "non singlet" since, in a more general situation, differences of this type transform as the adjoint representation of the flavour group. The $Q^2$ dependence implied by QCD can be simply expressed in terms of moments of parton densities. We therefore define *

$$M_n^{\mathrm{NS}}(t) = \int_0^1 \mathrm{d}x\, x^{n-1}\, q^{\mathrm{NS}}(x, t) \,. \tag{11}$$

As is well known [4,5] the predicted $t$ dependence of moments is of the form

$$M_n^{\mathrm{NS}}(t) = M_n^{\mathrm{NS}}(0) \left[\frac{\alpha}{\alpha(t)}\right]^{A_n^{\mathrm{NS}}/2\pi b} \,. \tag{12}$$

* In the following unless explicitly stated we restrict to moments with $n > 1$ in order to have convergence at $x = 0$ in moments integrals.

Here $\alpha(t)$ is the running coupling constant of QCD ($\alpha = g^2/4\pi$ in terms of the quark gluon vertex). In the leading logarithmic approximation $\alpha(t)$ is of the form [4,5]

$$\frac{\alpha}{\alpha(t)} = 1 + b\alpha t \,, \tag{13}$$

with $\alpha = \alpha(0)$ and $b$ a constant which is given in general by

$$b = \frac{11\, C_2(\mathrm{G}) - 4\, T(\mathrm{R})}{12\,\pi} \,, \tag{14}$$

where $C_2(\mathrm{G})$ and $T(\mathrm{R})$ are Casimir operators for the adjoint representation G of the color group and for the representation R of the fermions respectively. In the case of $\mathrm{SU}(N)_{\mathrm{color}}$ with $f$ flavours, they are given by

$$C_2(\mathrm{G}) = \frac{1}{N^2 - 1} \sum_{a,b,c} c_{abc}\, c_{abc} = N \,, \tag{15}$$

$$T(\mathrm{R})\, \delta_{ab} = \mathrm{Tr}(t^a t^b) = \tfrac{1}{2} f \delta_{ab} \,. \tag{16}$$

For the validity of eq. (13) it is necessary that $Q_0^2$, in eq. (10) be chosen large enough to make $\alpha(0)/\pi$ sufficiently small for a lowest formula to apply. Finally $A_n^{\mathrm{NS}}$ are a set of constants independent of $\alpha$ which we shall study in the following.

It is immediately seen that, for each $n$, eq. (12) is the solution of the differential equation

$$\frac{\mathrm{d}M_n^{\mathrm{NS}}(t)}{\mathrm{d}t} = \frac{\alpha(t)}{2\pi} A_n^{\mathrm{NS}}\, M_n^{\mathrm{NS}}(t) \,, \tag{17}$$

with assigned initial value $M_n^{\mathrm{NS}} = M_n^{\mathrm{NS}}(0)$ at $t = 0$. In turn, for any $n$ (sufficiently large) the whole set of eqs. (17) is equivalent to the following master equation for the densities:

$$\frac{\mathrm{d}q^{\mathrm{NS}}(x,t)}{\mathrm{d}t} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y}\, q^{\mathrm{NS}}(y,t)\, P\!\left(\frac{x}{y}\right) , \tag{18}$$

provides that

$$\int_0^1 \mathrm{d}z\, z^{n-1}\, P(z) = A_n^{\mathrm{NS}} \,. \tag{19}$$

This is easily been by multiplying both sides of eq. (18) by $x^{n-1}$, integrating from zero to one and interchanging the order of integration on the right-hand side. Eq. (18) is our starting point since it allows a simple physical interpretation of the function $P(z)$ which we shall use in the following for its direct computation (and of other similar generating functions).

It is convenient to rewrite eq. (18) in the form

$$q^{\rm NS}(x, t) + {\rm d}q^{\rm NS}(x, t) = \int_0^1 {\rm d}y \int_0^1 {\rm d}z\, \delta(zy - x)\, q^{\rm NS}(y, t) \left[\delta(z-1) + \frac{\alpha}{2\pi} P(z)\, {\rm d}t\right] . \tag{20}$$

The meaning of this equation is clear. Given a quark with momentum $y$ there is a chance that is radiates a gluon, thus reducing its energy from $y$ to $x$. The quantity

$$\mathcal{P}_{\rm qq} + {\rm d}\mathcal{P}_{\rm qq} = \delta(z-1) + \frac{\alpha}{2\pi} P(z)\, {\rm d}t \tag{21}$$

is the probability density of finding, inside a quark, another quark with fraction $z$ of the parent momentum. The change with $t$ of this probability produces the variation of the quark distribution function. Thus $P(z)\ \alpha/2\pi$ is the variation per unit $t$ at order $\alpha$ of the probability density of finding inside a quark another quark with fraction $z$ of the parent momentum.

We now drop the restriction to one flavour and to non-singlet densities. In parton language a singlet density is in general a combination of the sum of all quark and antiquark densities and of the gluon density inside the proton. We therefore introduce $G(x, t)$ as the density of gluons (summed over colors) inside the proton in the $p_\infty$ frame. We can now directly write down the integro-differential equations that describe the $Q^2$ dependence in the general case. They are

$$\frac{{\rm d}q^i(x, t)}{{\rm d}t} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{{\rm d}y}{y} \left[ \sum_{j=1}^{2f} q^j(y, t)\, P_{{\rm q}^i{\rm q}^j}\left(\frac{x}{y}\right) + G(y, t)\, P_{{\rm q}^i{\rm G}}\left(\frac{x}{y}\right)\right] , \tag{22}$$

$$\frac{{\rm d}G(x, t)}{{\rm d}t} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{{\rm d}y}{y} \left[ \sum_{j=1}^{2f} q^j(y, t)\, P_{{\rm Gq}^j}\left(\frac{x}{y}\right) + G(y, t)\, P_{\rm GG}\left(\frac{x}{y}\right)\right] . \tag{23}$$

Here the indices $i$ and $j$ run over quarks and antiquarks of all flavours. The number of quarks as seen by the current changes by two mechanisms: a quark originally at higher energy may loose momentum by radiating a gluon, or, a gluon inside the proton may produce a quark-antiquark pair. Similarly the number of gluons changes because a quark may radiate a gluon or because a gluon may split into a quark-antiquark pair or into two gluons. This last possibility is typical of non-Abelian gauge theories where a three gluon vertex exists to order $g$ (while the four gluon vertex is of order $g^2$).

Some properties of the functions $P(z)$ appearing in eqs. (22), (23) are immediately derived from the fact that color and flavour commute. First $P_{{\rm q}^i{\rm q}^j}$ is diagonal in quark indices because a gluon is emitted without flavour exchange,

$$P_{{\rm q}^i{\rm q}^j} = \delta_{ij}\, P_{\rm qq} \; . \tag{24}$$

Moreover, when we neglect all masses, the probability of emitting a gluon is the same for all flavours,

$$P_{\mathrm{Gq}^i} = P_{\mathrm{Gq}} \qquad \text{(independent of } i\text{)} . \tag{25}$$

Finally a gluon creates a massless quark-antiquark pair with equal probability for all flavours. Thus,

$$P_{\mathrm{q}^i\mathrm{G}} = P_{\mathrm{qG}} \qquad \text{(independent of } i\text{)} . \tag{26}$$

Therefore we can rewrite eqs. (22), (23) in the simpler form

$$\frac{\mathrm{d}q^i(x,t)}{\mathrm{d}t} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[ q^i(y,t)\, P_{\mathrm{qq}}\left(\frac{x}{y}\right) + G(y,t)\, P_{\mathrm{qG}}\left(\frac{x}{y}\right) \right], \tag{27}$$

$$\frac{\mathrm{d}G(x,t)}{\mathrm{d}t} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[ \sum_{i=1}^{2f} q^i(y,t)\, P_{\mathrm{Gq}}\left(\frac{x}{y}\right) + G(y,t)\, P_{\mathrm{GG}}\left(\frac{x}{y}\right) \right]. \tag{28}$$

By summing eq. (27) over $i = 1, ..., 2f$, we obtain

$$\frac{\mathrm{d}\Sigma_{i=1}^{2f} q^i(x,t)}{\mathrm{d}t} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[ \sum_{i=1}^{2f} q^i(y,t)\, P_{\mathrm{qq}}\left(\frac{x}{y}\right) + G(y,t)\, 2f\, P_{\mathrm{qG}}\left(\frac{x}{y}\right) \right]. \tag{29}$$

It is precisely the matrix

$$\int_0^1 \mathrm{d}z\, z^{n-1} \begin{bmatrix} P_{\mathrm{qq}}(z) & 2f\, P_{\mathrm{qG}}(z) \\ P_{\mathrm{Gq}}(z) & P_{\mathrm{GG}}(z) \end{bmatrix} \equiv \begin{bmatrix} A_n^{\mathrm{NS}} & 4\, T(\mathrm{R})\, A_n^{\mathrm{qG}} \\ A_n^{\mathrm{Gq}} & A_n^{\mathrm{GG}} \end{bmatrix}, \tag{30}$$

which gives the logarithmic exponents for each $n$ as given in the literature [4,5]. For each value $n$ this matrix must be diagonalized in order to find the eigenvectors and the eigenvalues of the $Q^2$ evolution equations.

By subtracting the derivatives for a quark and an antiquark (or for two quarks) we immediately recover the equation for the non-singlet case, eq. (18),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[q^i(x,t) - q^j(x,t)\right]$$

$$= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[q^i(y,t) - q^j(y,t)\right] P_{\mathrm{qq}}\left(\frac{x}{y}\right). \tag{31}$$

We thus reproduce the result that $P_{\mathrm{qq}}$ is the same function for singlet quarks and for all types of non-singlet quarks.

The non-diagonal functions $(\alpha/2\pi)P_{\mathrm{Gq}}(z)$ and $(\alpha/2\pi)P_{\mathrm{qG}}(z)$ can be directly interpreted as probability densities $(\alpha/2\pi)P_{\mathrm{Gq}}(z)$ is the probability density per unit $t$ at order $\alpha$ of finding a gluon inside a quark (or an antiquark) with fraction $z$ of the longitudinal momentum of the parent quark. $(\alpha/2\pi)P_{\mathrm{qG}}(z)$ is the probability density per unit $t$ at order $\alpha$ of finding a quark (or an antiquark) inside a gluon with fraction $z$ of the longitudinal momentum of the parent gluon. For the diagonal functions $(\alpha/2\pi)P_{qq}(z)$ and $(\alpha/2\pi)P_{GG}(z)$ the actual probabilities also involve delta function singularities at $z = 1$. The probability densities are given in these cases by eq. (21) and by

$$\mathcal{P}_{GG} + d\mathcal{P}_{GG} = \delta(z-1) + \frac{\alpha}{2\pi} P_{GG}(z)\, dt\,, \tag{32}$$

Therefore it is only at $z < 1$ that $(\alpha/2\pi)P_{qq}(z)$ and $(\alpha/2\pi)P_{GG}(z)$ are probability densities (for example they must be positive definite). Consider for example $(\alpha/2\pi)P_{qq}(z)$. Since the total number of quarks minus antiquarks is conserved, the probability of finding a quark in a quark, integrated over all values of $z$, must add up to one. It follows that the integrated correction of order $\alpha$ must be zero

$$\int_0^1 dz\, P_{qq}(z) = 0\,, \tag{33}$$

which is a condition on $A_1^{NS}$. This is the well known result that charges (i.e. current algebra sum rules) are protected against $Q^2$ corrections [4,5]. We see that the presence of $\delta$ function singularities destroys the positive definiteness of integrals of $P_{qq}$ and $P_{GG}$. In particular we shall see that the values of $A_n^{NS}$ are all negative for $n > 1$.

Momentum conservation in the vertices imposes further constraints on the $P$ functions. At $z < 1$ we have

$$\begin{aligned} P_{qq}(z) &= P_{Gq}(1-z)\,, \\ P_{qG}(z) &= P_{qG}(1-z) \qquad (z<1)\,, \\ P_{GG}(z) &= P_{GG}(1-z)\,. \end{aligned} \tag{34}$$

The above equations arise because when the quark radiates, it splits into a quark with fraction $z$ of its momentum plus a gluon with fraction $(1-z)$ etc. However at $z = 1$ the $\delta$ function singularities upset these relations. But it must remain true that

$$\begin{aligned} &\int_0^1 dz\, z\,[P_{qq}(z) + P_{Gq}(z)] = 0\,, \\ &\int_0^1 dz\, z\,[2f\, P_{qG}(z) + P_{GG}(z)] = 0\,, \end{aligned} \tag{35}$$

which guarantee that the total momentum of the proton (i.e. of all partons) is unchanged,

$$\frac{d}{dt}\int_0^1 dx\, x\left[\sum_{i=1}^{2f} q^i(x,t) + G(x,t)\right] = 0 \tag{36}$$

as can be seen from eqs. (28) and (29). In the following we shall use eqs. (33) and (35) to fix the $\delta$ function singularities at $z = 1$ of $P_{\mathrm{qq}}$ and $P_{\mathrm{GG}}$.

We can make the symmetry relations eqs. (34) hold for all values of $z$ if we treat the points $z = 0$ and $z = 1$ symmetrically i.e. if we artificially add (for the sake of this argument only) a $\delta$ singularity at $z = 0$ when the symmetry requires it. Note that moments with $n > 1$ are unaffected by a $\delta(z)$ term. The interest of this observation is that we have, for example,

$$\int_0^1 \mathrm{d}z\, z^{n-1} P_{\mathrm{qq}}(z) = A_n^{\mathrm{NS}} = \int_0^1 \mathrm{d}z\, z^{n-1}\, P_{\mathrm{Gq}}(1-z)$$

$$= \int_0^1 \mathrm{d}z\, (1-z)^{n-1}\, P_{\mathrm{Gq}}(z) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} A_{k+1}^{\mathrm{Gq}} \,. \tag{37}$$

Through eq. (33) the previous sum rule would imply for $n = 1$ that $A_1^{\mathrm{Gq}} = 0$. This equality is not true but is the only consequence of having artificially added a $\delta(z)$ term to $P_{\mathrm{Gq}}$. We can however use it to eliminate $A_1^{\mathrm{Gq}}$ from eq. (37) and write a sum rule for the logarithmic exponents with $n > 1$ which reads

$$A_n^{\mathrm{NS}} = \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} A_{k+1}^{\mathrm{Gq}} \,. \tag{38}$$

This sum rule must be (and is) satisfied by the calculated logarithmic exponents and offers a general test of the above physical interpretation. The origin of eq. (38) in the light cone language is not easy to disentangle. Another similar sum rule is obtained for the combination

$$B_n = 2f A_n^{\mathrm{qG}} + A_n^{\mathrm{GG}} \tag{39}$$

by using the symmetry relation eqs. (34) and the fact that $B_2 = 0$ as stated in eqs. (35). This sum rule reads

$$B_n = \sum_{k=2}^{n-1} (-1)^k \binom{n-1}{k} B_{k+1} \,,$$

or equivalently,

$$[1 + (-1)^n] B_n = \sum_{j=3}^{n-1} (-1)^{j-1} \binom{n-1}{j-1} B_j \,. \tag{40}$$

This relation implies that the $B_n$'s for even $n$ are determined by the $B_n$'s for odd $n$.

We conclude this section by observing that master equations of the form of eqs. (27), (28) are valid in any renormalizable theory at lowest order in perturbation theory. Therefore they are reliable provided the coupling constant is sufficiently small. For example they are valid in quantum electrodynamics of electrons and photons. In this case $\mathcal{P}_{GG} \to \mathcal{P}_{\gamma\gamma} \sim \delta(z-1)$ because of the absence of a $3\gamma$ coupling. The replacement of $\alpha$ by $\alpha(t)$ in QCD guarantees that for sufficiently large $t$ the master equations will eventually become reliable. These master equations can provide a useful insight in connection with problems that cannot be approached by light-cone expansion.

## 4. Calculation of logarithmic exponents. Spin averaged case

In this section we show that the functions $P(z)$ introduced in the previous section can be directly computed from the simple knowledge of the basic vertices of QCD. The method used is an extension of the von Weizsacker-Williams result in quantum electrodynamics [8]. In that case the equivalent number of photons inside an electron with fraction $z$ of the electron momentum is evaluated to order $\alpha$ and contains a factor of $\ln E/m_e$, which plays the same role as $t = \ln Q^2/Q_0^2$ in our case.

We first compute the functions $P(z)$ at $z < 1$ while we shall deal, at the end, with $\delta$ function singularities at $z = 1$. We want to evaluate the probability of finding a particle B inside a particle A with fraction $z$ of the longitudinal momentum of A in the $p_\infty$ frame to lowest order in $\alpha$:

$$\mathrm{d}\mathcal{P}_{BA}(z)\,\mathrm{d}z = \frac{\alpha}{2\pi} P_{BA}(z)\,\mathrm{d}z\,\mathrm{d}t\,. \tag{41}$$

Let C be the third particle in the bare vertex where A and B appear. We can identify the above probability by comparing the cross sections for the two processes in fig. 1, where D is a given particle and f any final state. We define the general $S$-matrix element as

$$S_{ij} - \delta_{ij} = 2\pi i\delta(E_j - E_i)\,M_{ij} \prod_k (2E_k)^{-1/2}\,, \tag{42}$$

where the index $k$ runs over all external particles. Although by no means necessary, we find it particularly useful to phrase our calculation in terms of the "old" perturbation theory which is best suited for a discussion of leading terms in the $p_\infty$ frame. The contribution to $M_{ij}$ in eq. (42) of a given intermediate state B to the process in

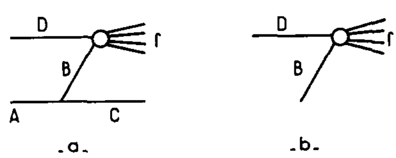


Fig. 1. (a) Contribution of the B intermediate state to the process A + D → C + f. (b) The process B + D → f.

fig. 1a can be written as

$$M_{\mathrm{A+D\to C+f}} = g^2 \frac{V_{\mathrm{A\to B+C}}\, V_{\mathrm{B+D\to f}}}{(2E_\mathrm{B})(E_\mathrm{B}+E_\mathrm{C}-E_\mathrm{A})}\,, \tag{43}$$

where $V_{ij}$ is the invariant matrix element of the interactions (with the factors $(2E_k)^{-1/2}$ removed). Similarly in the case of fig. 1b we have

$$M_{\mathrm{B+D\to f}} = g\, V_{\mathrm{B+D\to f}}\,. \tag{44}$$

The contribution of a given intermediate state B to $\mathrm{d}\sigma_a$ and $\mathrm{d}\sigma_b$ is therefore given by

$$\mathrm{d}\sigma_a = \frac{g^4}{8E_\mathrm{A}E_\mathrm{D}} \frac{|V_{\mathrm{A\to B+C}}|^2 |V_{\mathrm{B+D\to f}}|^2}{(2E_\mathrm{B})^2(E_\mathrm{B}+E_\mathrm{C}-E_\mathrm{A})^2}$$

$$\times (2\pi)^2\, \delta^4(k_\mathrm{A}+k_\mathrm{D}-k_\mathrm{C}-k_\mathrm{f}) \frac{\mathrm{d}^3 k_\mathrm{C}}{(2\pi)^3(2E_\mathrm{C})} \prod_f \frac{\mathrm{d}^3 p_\mathrm{f}}{(2\pi)^3(2E_\mathrm{f})}\,, \tag{45}$$

$$\mathrm{d}\sigma_b = \frac{g^2}{8E_\mathrm{B}E_\mathrm{D}} |V_{\mathrm{B+D\to f}}|^2 (2\pi)^4 \delta^4(k_\mathrm{B}+k_\mathrm{D}-k_\mathrm{f}) \prod_\mathrm{f} \frac{\mathrm{d}^3 p_\mathrm{f}}{(2\pi)^3(2E_\mathrm{f})}\,. \tag{46}$$

The two processes are related by

$$\mathrm{d}\sigma_a = \mathrm{d}\mathcal{P}_{\mathrm{BA}}(z)\, \mathrm{d}z\, \mathrm{d}\sigma_b\,. \tag{47}$$

From eq. (47), by the comparison of eqs. (45) and (46) we obtain

$$\mathrm{d}\mathcal{P}_{\mathrm{BA}}(z)\mathrm{d}z = \frac{E_\mathrm{B}}{E_\mathrm{A}} \frac{g^2|V_{\mathrm{A\to B+C}}|^2}{(2E_\mathrm{B})^2(E_\mathrm{B}+E_\mathrm{C}-E_\mathrm{A})^2} \frac{\mathrm{d}^3 k_\mathrm{C}}{(2\pi)^3(2E_\mathrm{C})}\,, \tag{48}$$

where all masses being neglected:

$$k_\mathrm{A} = (P; P, \mathbf{0})\,,$$

$$k_\mathrm{B} = \left(zP + \frac{p_\perp^2}{2zP}; zP, \boldsymbol{p}_\perp\right),$$

$$k_\mathrm{C} = \left((1-z)P + \frac{p_\perp^2}{2(1-z)P}; (1-z)P, -\boldsymbol{p}_\perp\right). \tag{49}$$

We thus have

$$(2E_\mathrm{B})^2(E_\mathrm{B}+E_\mathrm{C}-E_\mathrm{A})^2 = \frac{(p_\perp^2)^2}{(1-z)^2}\,, \tag{50}$$

$$\frac{\mathrm{d}^3 k_\mathrm{C}}{(2\pi)^3(2E_\mathrm{C})} = \frac{\mathrm{d}z\, \mathrm{d}p_\perp^2}{16\pi^2(1-z)}\,. \tag{51}$$

The result is

$$\mathrm{d}\mathcal{P}_{\mathrm{BA}}(z) = \frac{\alpha}{2\pi}\frac{z(1-z)}{2}\overline{\sum_{\text{spins}}}\frac{|V_{\mathrm{A}\to\mathrm{B+C}}|^2}{p_\perp^2}\,\mathrm{d}\ln p_\perp^2\,, \tag{52}$$

where a sum over the spins of B and C and an average over the spin of A is indicated (if the case).

In the case of interst $|V|^2$ vanishes linearly in $p_\perp^2$ so that the ratio $|V|^2/p_\perp^2$ in eq. (52) is finite at $p_\perp^2 = 0$. Moreover for a virtual mass $-Q^2$ for particle D, the integral in $p_\perp^2$ has an upper limit of order $Q^2$, so that, at the leading logarithmic approximation, $\mathrm{d}\ln p_\perp^2$ can be directly interpreted as $\mathrm{d}t$. We thus finally obtain, by comparing eq. (52) with eq. (41),

$$P_{\mathrm{BA}}(z) = \tfrac{1}{2}\,z(1-z)\overline{\sum_{\text{spins}}}\frac{|V_{\mathrm{A}\to\mathrm{B+C}}|^2}{p_\perp^2} \qquad (z<1)\,, \tag{53}$$

an expression which only depends on the vertex ABC. In particular when the spin sum is symmetric we clearly have

$$P_{\mathrm{CA}}(z) = P_{\mathrm{BA}}(1-z) \qquad (z<1) \tag{54}$$

in agreement with eqs. (34).

Let us now specialize to the quark gluon vertex in fig. 2 in order to evaluate $P_{\mathrm{Gq}}(z)$. In this case,

$$\overline{\sum_{\text{spins}}}\,|V_{q\to\mathrm{Gq}}|^2 = \tfrac{1}{2}C_2(\mathrm{R})\,\mathrm{Tr}(\not{k}_{\mathrm{C}}\gamma_\mu\not{k}_{\mathrm{A}}\gamma_\nu)\overline{\sum_{\text{pol}}}\epsilon^{*\mu}\epsilon^{\nu}\,, \tag{55}$$

where the factor of $\frac{1}{2}$ derives from the average over the initial quark spin, and $C_2(\mathrm{R})$ defined by

$$C_2(\mathrm{R}) = \frac{1}{N}\sum_a t^a t^a = \frac{N^2-1}{2N} \tag{56}$$

arises from the sum and average over the final and initial states in color space. Care must be taken so that only physical transverse gluon states are included in the sum, and we therefore write

$$\sum_{\text{pol}}\epsilon^*\epsilon \to \delta^{ij} - \frac{k_{\mathrm{B}}^i k_{\mathrm{B}}^j}{k_{\mathrm{B}}^2} \qquad (i,j=1,2,3)\,. \tag{57}$$

One obtains

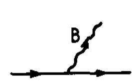


Fig. 2. The quark gluon vertex which determines $P_{\mathrm{Gq}}$ and $P_{\mathrm{qq}}$. The form of the vertex is $ig\bar{q}_{\mathrm{C}}\gamma^\mu t^a q_{\mathrm{A}} B_\mu^a$ with $\mathrm{Tr}\,t^a t^b = \frac{1}{2}\delta^{ab}$.

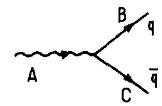


Fig. 3. The annihilation vertex of a gluon into a quark-antiquark pair which fixes $P_{qG}$.

$$\overline{\sum_{\rm pol}} |V_{q\to G+q}|^2 = \frac{2p_\perp^2}{z(1-z)} \frac{1+(1-z)^2}{z} C_2(R)\,. \tag{58}$$

We can thus state the result from eqs. (53), (58),

$$P_{Gq}(z) = C_2(R)\,\frac{1+(1-z)^2}{z}\,, \tag{59}$$

which holds at all $z$, since we are dealing with a non-diagonal density. From the last equation, by using the symmetry relations eqs. (34) we also obtain

$$P_{qq}(z) = C_2(R)\,\frac{1+z^2}{1-z} \qquad (z<1) \tag{60}$$

We now calculate $P_{qG}$ from the vertex in fig. 3. Since $P_{qG}$ is proportional to the probability density of finding inside a gluon (averaged over colors) a quark (or an antiquark) of given flavour and of any color, in this case the sum and average in color space simply bring in a factor of $\frac{1}{2}$. We thus have

$$\overline{\sum_{\rm spins}} |V_{G\to q+\bar q}|^2 = \tfrac{1}{2}\,{\rm Tr}(\not k_C \gamma_\mu \not k_B \gamma_\nu)\, \tfrac{1}{2} \sum_{\rm pol} \epsilon^{*\mu}\epsilon^\nu\,, \tag{61}$$

which, recalling eq. (57), gives

$$\sum_{\rm spins} |V_{G\to q+\bar q}|^2 = p_\perp^2\left(\frac{1-z}{z} + \frac{z}{1-z}\right).$$

From eq. (53) we finally obtain

$$P_{qG}(z) = \tfrac{1}{2}(z^2 + (1-z)^2)\,. \tag{62}$$

The symmetry under the change of $z$ into $(1-z)$ is expected because $P_{qG}(z) = P_{\bar q G}(1-z) = P_{qG}(1-z)$, by eqs. (26) and (34).

We now turn to the three-gluon vertex in fig. 4 which determines $P_{GG}(z)$. The

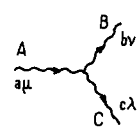


Fig. 4. The three-gluon vertex relevant to determine $P_{GG}$. It is equal to

$$-igc_{abc}[g_{\nu\mu}(k_A + k_B)_\lambda - g_{\mu\lambda}(k_C + k_A)_\nu + g_{\lambda\nu}(k_C - k_B)_\mu]\,.$$

amplitude is given by

$$V_{G\to G+G} = -c_{abc}\{-[(k_A + k_C)\,\epsilon_B^{*b}](\epsilon_A^a\,\epsilon_C^{*c}) + [(k_C - k_B)\epsilon_A^a](\epsilon_C^{*c}\epsilon_B^{*b}) + [(k_A + k_B)\epsilon_C^{*c}](\epsilon_A^a\,\epsilon_B^{*b})\}\,. \tag{63}$$

It is only a matter of algebra to derive the result* (recall eq. (15))

$$\overline{\sum_{\text{spins}}}\,|V_{G\to G+G}|^2 = 4C_2(G)\frac{p_\perp^2}{z(1-z)}\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]. \tag{64}$$

Through eq. (53) this equation leads to

$$P_{GG}(z) = 2C_2(G)\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] \qquad (z<1)\,. \tag{65}$$

Note the obvious symmetry for $z$ into $(1-z)$ in agreement with eq. (34).

We now complete the determination of $P_{qq}(z)$ and $P_{GG}(z)$ by fixing their behaviour at $z = 1$. Note that all moments of these functions as given by eqs. (60) and (65) would be divergent at $z = 1$. We therefore start by regularizing the factor $(1-z)^{-1}$ by reinterpreting it as a distribution $(1-z)_+^{-1}$ defined in the following way:

$$\int_0^1 \frac{dz\, f(z)}{(1-z)_+} \equiv \int_0^1 dz\,\frac{f(z)-f(1)}{1-z} = \int_0^1 dz\,\ln(1-z)\,\frac{d}{dz}f(z)\,, \tag{66}$$

with $f(z)$ being any test function which is sufficiently regular at the end points. In particular we have

$$\int_0^1 dz\,\frac{1}{(1-z)_+} = 0\,. \tag{67}$$

We then add, to $P_{qq}(z)$ and $P_{GG}(z)$, a $\delta(z-1)$ function with the coefficient determined by the constraints in eqs. (33) and (35). We thus find

$$P_{qq}(z) = C_2(R)\left[\frac{1+z^2}{(1-z)_+} + \tfrac{3}{2}\,\delta(z-1)\right], \tag{68}$$

$$P_{GG}(z) = 2C_2(G)\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \left(\frac{11}{12} - \frac{1}{3}\frac{T(R)}{C_2(G)}\right)\delta(z-1)\right]. \tag{69}$$

It is now straightforward to evaluate the moments of the $P(z)$ functions, which give the set of constants $A_n$ according to eq. (30). We first evaluate the moments of $(1-z)_+^{-1}$;

* We shall study this amplitude in all details when dealing with the polarized case in sect. 5.

$$\int_0^1 dz \frac{z^{n-1}}{(1-z)_+} \equiv \int_0^1 dz \frac{z^{n-1}-1}{1-z} = \sum_{j=1}^{n-1} \frac{(-1)^j}{j} \binom{n-1}{j}$$

$$= -\sum_{j=1}^{n-1} \frac{1}{j}, \tag{70}$$

where the last equality can be easily proved by induction. We finally obtain

$$\int_0^1 dz\, z^{n-1} P_{\rm qq}(z) \equiv A_n^{\rm NS} = C_2({\rm R}) \left[ -\tfrac{1}{2} + \frac{1}{n(n+1)} - 2 \sum_{j=2}^{n} \frac{1}{j} \right], \tag{71}$$

$$\int_0^1 dz\, z^{n-1} P_{\rm Gq}(z) \equiv A_n^{\rm Gq} = C_2({\rm R}) \frac{2+n+n^2}{n(n^2-1)}, \tag{72}$$

$$2f \int_0^1 dz\, z^{n-1} P_{\rm qG}(z) \equiv 4T({\rm R}) A_n^{\rm qG} = 2T({\rm R}) \frac{2+n+n^2}{n(n+1)(n+2)}, \tag{73}$$

$$\int_0^1 dz\, z^{n-1} P_{\rm GG}(z) \equiv A_n^{\rm GG}$$

$$= C_2({\rm G}) \left[ -\frac{1}{6} + \frac{2}{n(n-1)} + \frac{2}{(n+1)(n+2)} - 2 \sum_{j=2}^{n} \frac{1}{j} - \frac{2}{3} \frac{T({\rm R})}{C_2({\rm G})} \right]. \tag{74}$$

This set of logarithmic exponents, taking eq. (12) into account, is seen to coincide with the results of refs. [4,5].

## 5. Calculation of logarithmic exponents. Spin-dependent case

In this section we shall consider the $Q^2$ dependence of quark densities with given helicity which are relevant for scaling breaking effects in deep inelastic scattering on polarized targets.

The experience accumulated over the previous sections allows us to directly write down the general master equations which are

$$\frac{d}{dt} q_+^i(x,t) = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ q_+^i(y,t) P_{\rm q_+q_+}\left(\frac{x}{y}\right) + q_-^i(y,t) P_{\rm q_+q_-}\left(\frac{x}{y}\right) \right.$$

$$\left. + G_+(y,t) P_{\rm q_+G_+}\left(\frac{x}{y}\right) + G_-(y,t) P_{\rm q_+G_-}\left(\frac{x}{y}\right) \right],$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q^i_-(x,t) = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[ q^i_+(y,t)\, P_{q_-q_+}\left(\frac{x}{y}\right) + q^i_-(y,t)\, P_{q_-q_+}\left(\frac{x}{y}\right) \right.$$

$$\left. + G_+(y,t) P_{q_-G_+}\left(\frac{x}{y}\right) + G_-(y,t)\, P_{q_-G_-}\left(\frac{x}{y}\right)\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}t} G_+(x,t) = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[ \sum_{i=1}^{2f} q^i_+(y,t)\, P_{G_+q_+}\left(\frac{x}{y}\right) + \sum_{i=1}^{2f} q^i_-(y,t)\, P_{G_+q_-}\left(\frac{x}{y}\right) \right.$$

$$\left. + G_+(y,t)\, P_{G_+G_+}\left(\frac{x}{y}\right) + G_-(y,t)\, P_{G_+G_-}\left(\frac{x}{y}\right)\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}t} G_-(x,t) = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[ \sum_{i=1}^{2f} q^i_+(y,t)\, P_{G_-q_+}\left(\frac{x}{y}\right) + \sum_{i=1}^{2f} q^i_-(y,t)\, P_{G_-q_-}\left(\frac{x}{y}\right) \right.$$

$$\left. + G_+(y,t)\, P_{G_-G_+}\left(\frac{x}{y}\right) + G_-(y,t)\, P_{G_-G_-}\left(\frac{x}{y}\right)\right]. \qquad (75)$$

The previous set of equations can immediately be simplified by observing that parity conservation in QCD implies the relations:

$$P_{A_+B_\pm}(z) = P_{A_-B_\mp}(z) \qquad (76)$$

for any A and B. These relations in turn imply, (when used in eqs. (75)) that the sums $q^i_+ + q^i_- = q^i$ and $G_+ + G_- = G$ and the differences

$$\Delta q^i = q^i_+ - q^i\ , \qquad (77)$$

$$\Delta G = G_+ - G_-\ , \qquad (78)$$

evolve separately. For the sums we recover eqs. (27) and (28) with the obvious identification that

$$P_{AB} = P_{A_+B_+} + P_{A_-B_+}\ . \qquad (79)$$

It is convenient to define

$$\Delta P_{AB} = P_{A_+B_+} - P_{A_-B_+}\ . \qquad (80)$$

We then derive, from eqs. (75), the master equations for the differences

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta q^i(x,t) = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[\Delta q^i(y,t)\, \Delta P_{qq}\left(\frac{x}{y}\right) + \Delta G(y,t)\, \Delta P_{qG}\left(\frac{x}{y}\right)\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta G(x,t) = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \left[\sum_{i=1}^{2f} \Delta q^i(y,t)\, \Delta P_{Gq}\left(\frac{x}{y}\right) + \Delta G(y,t)\, \Delta P_{GG}\left(\frac{x}{y}\right)\right]. \qquad (81)$$

In the simplest case of non-singlet quark densities, eqs. (81) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta q^{\mathrm{NS}}(x, t) = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} \Delta q^{\mathrm{NS}}(y, t)\, \Delta P_{\mathrm{qq}}\left(\frac{x}{y}\right). \tag{82}$$

For example this allow us to conclude from eq. (8) that the non-singlet part of the structure function $G_1$ is an eigenvector of the $Q^2$ dependence. Actually we can immediately obtain a much stronger result in that, when masses are neglected, the vector quark-gluon coupling in fig. 2 is helicity conserving. Therefore we can directly conclude that

$$P_{\mathrm{q}_-\mathrm{q}_+}(z) = 0 \,,$$

$$P_{\mathrm{q}_+\mathrm{q}_+}(z) = P_{\mathrm{qq}}(z) = \Delta P_{\mathrm{qq}}(z) = C_2(\mathrm{R})\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(z-1)\right]. \tag{83}$$

We thus obtain that the moments of $q^{\mathrm{NS}}(x, t)$ and of $\Delta q^{\mathrm{NS}}(x, t)$ (or equivalently of the non-singlet parts of $F_1$ and $G_1$) evolve in $Q^2$ with the same logarithmic exponents.

The same quark-gluon vertex of fig. 2 also determines $P_{\mathrm{G}_+\mathrm{q}_+}(z)$ and $P_{\mathrm{G}_-\mathrm{q}_+}(z)$. Given the gluon momentum in the form (see eqs. (49))

$$q = \left(zP + \frac{p_\perp^2}{2zP}\,;\, zP, p_x, p_y\right) \tag{84}$$

for the corresponding polarization four-vector for positive and negative helicity we can approximately set

$$\epsilon_\pm \simeq \left(0; -\frac{p_x \pm i p_y}{\sqrt{2}zP}\,, \sqrt{\tfrac{1}{2}}\,, \pm i\sqrt{\tfrac{1}{2}}\right), \tag{85}$$

where only the leading term for each component was kept, which is sufficient for our purposes. For the invariant vertex $V_{\mathrm{q}_+\to\mathrm{G}_\pm + q}$ (see eq. (55) for comparison; a sum over the final quark spin is understood) one has

$$|V_{\mathrm{q}_+\to\mathrm{G}_\pm\mathrm{q}}|^2 = C_2(\mathrm{R})\,\mathrm{Tr}\left(\not{k}_{\mathrm{C}}\gamma_\mu \not{k}_{\mathrm{A}}\gamma_\nu \frac{1+\gamma_5}{2}\right) \epsilon_\pm^{*\mu}\epsilon_\pm^\nu$$

$$= \frac{p_\perp^2}{2z(1-z)}\frac{1}{z}\left[(2-2z+z^2) \pm z(2-z)\right].$$

By using eq. (53) we thus obtain

$$P_{\mathrm{G}_+\mathrm{q}_+}(z) = C_2(\mathrm{R})\frac{1}{z}\,, \tag{86}$$

$$P_{\mathrm{G}_-\mathrm{q}_+}(z) = C_2(\mathrm{R})\frac{(1-z)^2}{z}\,, \tag{87}$$

and consequently,

$$\Delta P_{\mathrm{Gq}}(z) = C_2(\mathrm{R})\, \frac{1-(1-z)^2}{z}\,, \tag{88}$$

while the sum of the two reproduces eq. (59) for $P_{\mathrm{Gq}}(z)$ in agreement with eq. (79).

We now evaluate $P_{\mathrm{q_+G_+}}(z)$ and $P_{\mathrm{q_-G_+}}(z)$ from the annihilation vertex in fig. 3. A sum over the final antiquark spin being understood, the invariant vertex gives

$$|V_{\mathrm{G_+\to q_\pm \bar{q}}}|^2 = \tfrac{1}{2}\,\mathrm{Tr}\left(\not{k}_{\mathrm{C}}\gamma_\mu \not{k}_{\mathrm{B}}\gamma_\nu\, \frac{1\pm\gamma_5}{2}\right) \epsilon_+^{*\mu}\epsilon_+^{\nu}$$

$$= \frac{p_\perp^2}{z(1-z)}\,[(z^2+(1-z)^2) \pm (z^2-(1-z)^2)]\,. \tag{89}$$

(Note that $\epsilon_\mu$ now refers to the initial gluon so that it is given by eq. (85) with $\boldsymbol{p}_\perp = 0$). From eqs. (53) and (89) we obtain

$$P_{\mathrm{q_+G_+}}(z) = \tfrac{1}{2}z^2\,, \tag{90}$$

$$P_{\mathrm{q_-G_+}}(z) = \tfrac{1}{2}(1-z)^2\,, \tag{91}$$

$$\Delta P_{\mathrm{qG}}(z) = \tfrac{1}{2}[z^2-(1-z)^2]\,, \tag{92}$$

while the sum reproduces $P_{\mathrm{qG}}$ in eq. (62). Last but not least, we must now evaluate $P_{\mathrm{G_+G_+}}$ and $P_{\mathrm{G_-G_+}}$ from the three gluon vertex in fig. 4. The invariant amplitude was written down in eq. (63). When the impulses of the gluons are specified as in eq. (49) the polarization vectors are

$$\epsilon_{+\mathrm{A}} = \sqrt{\tfrac{1}{2}}(0;0,1,i)\,,$$

$$\epsilon_{\pm\mathrm{B}} = \sqrt{\tfrac{1}{2}}\left(0;-\frac{p_x \pm ip_y}{zP},1,\pm i\right), \tag{93}$$

$$\epsilon_{\pm\mathrm{C}} = \sqrt{\tfrac{1}{2}}\left(0;\frac{p_x \pm ip_y}{(1-z)P},1,\pm i\right). \tag{93}$$

It is more expedient to calculate the four helicity amplitudes first and to take the square modulus and sum over the polarizations of the gluon C at the end. A simple calculation leads to the following results for $V_{\mathrm{G_+\to G_\pm G_\pm}}$:

$$V_{+++} = -\sqrt{2}\,c_{abc}(p_x - ip_y)\left(\frac{1}{z}+\frac{1}{1-z}\right),$$

$$V_{+-+} = -\sqrt{2}\,c_{abc}(p_x + ip_y)\left(\frac{1}{z}-1\right),$$

$$V_{++-} = -\sqrt{2}\, c_{abc}(p_x + ip_y) \left(\frac{1}{1-z} - 1\right) ,$$

$$V_{+--} = 0 . \tag{94}$$

By using eq. (53), a simple algebra leads to

$$P_{G_+G_+}(z) = C_2(G)\,(1 + z^4)\left(\frac{1}{z} + \frac{1}{1-z}\right), \qquad (z < 1) , \tag{95}$$

$$P_{G_-G_+}(z) = C_2(G)\, \frac{(1-z)^3}{z} . \tag{96}$$

The behaviour at $z = 1$ of the diagonal density $P_{G_+G_+}(z)$ is immediately obtained from that of $P_{GG}(z)$ in eq. (69) by recalling eq. (79) (which is verified by eqs. (95) and (96) at $z < 1$). We thus obtain

$$P_{G_+G_+}(z) = C_2(G)\left[(1 + z^4)\left(\frac{1}{z} + \frac{1}{(1-z)_+}\right) + \left(\frac{11}{6} - \frac{2}{3}\frac{T(R)}{C_2(G)}\right)\delta(z-1)\right] , \tag{97}$$

$$\Delta P_{GG}(z) = C_2(G)\left[(1 + z^4)\left(\frac{1}{z} + \frac{1}{(1-z)_+}\right) - \frac{(1-z)^3}{z} + \left(\frac{11}{6} - \frac{2}{3}\frac{T(R)}{C_2(G)}\right)\delta(z-1)\right] \tag{98}$$

Collecting our results eqs. (83), (88), (92) and (98) we are now ready to compute the matrix of the logarithmic exponents

$$\int_0^1 \mathrm{d}z\, z^{n-1} \begin{bmatrix} \Delta P_{qq}(z) & 2f\Delta P_{qG}(z) \\ \Delta P_{Gq}(z) & \Delta P_{GG}(z) \end{bmatrix} = \begin{bmatrix} \widetilde{A}_n^{NS} & 4T(R)\widetilde{A}_n^{qG} \\ \widetilde{A}_n^{Gq} & \widetilde{A}_n^{GG} \end{bmatrix} , \tag{99}$$

which are

$$\widetilde{A}_n^{NS} = A_n^{NS} ,$$

$$\widetilde{A}_n = C_2(R)\, \frac{n+2}{n(n+1)} ,$$

$$4T(R)\,\widetilde{A}_n^{qG} = 2T(R)\, \frac{n-1}{n(n+1)} ,$$

$$\widetilde{A}_n^{GG} = C_2(G)\left[\frac{11}{6} - \frac{2}{3}\frac{T(R)}{C_2(G)} + \frac{2}{n} - \frac{4}{n+1} - 2\sum_{j=1}^{n-1}\frac{1}{j}\right]. \tag{100}$$

Taking eq. (12) into account, this set of values is in agreement with the result of refs. [6,7], obtained by the operator formalism.

## 6. Concluding remarks

We briefly comment on the reasons for the success of this simple method of calculating the logarithmic exponents and the limitations of the present approach. The essential point is the following. The exact renormalization group equations are given by

$$\left[\frac{\partial}{\partial t} - \beta(\alpha)\frac{\partial}{\partial\alpha} - \gamma_n(\alpha)\right] M_n(\alpha, t) = 0 \,. \tag{101}$$

We refer to the simplest case of a multiplicatively renormalizable operator, whose coefficient function is proportional to the moment $M_n$ of some structure function. At lowest order in $\alpha$ one has

$$\gamma_n(\alpha) \approx \frac{\alpha}{2\pi} A_n \,. \tag{102}$$

Eqs. (12) and (13) show the solution of eq. (101) to lowest order in $\alpha$. From eq. (17) we see that the solution of eq. (101) also satisfies the simpler equation to lowest order in $\alpha$,

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - \gamma_n(\alpha(t))\right] M_n \approx \left[\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\alpha(t)}{2\pi} A_n\right] M_n = 0 \,. \tag{103}$$

Thus, to lowest order the only effect of the $\beta$ function, which embodies all the intricacies of the coupling constant renormalization, is to replace $\alpha$ by the running coupling constant in the expression of $\gamma_n$. Therefore the lowest order master equations through this simple replacement become the solutions of this more sophisticated problem. In higher orders the effects of $\beta(\alpha)$ cannot be taken into account in this trivial way and the evaluation of the functions $P(z)$ would be more complicated.

Another feature which makes the calculations particularly simple is the direct and transparent connection of the relevant bilocal operators in terms of parton densities of given helicity in the $p_\infty$ frame. It would be interesting to investigate whether a similar approach can be applied to the computation of the logarithmic exponents of more complicated operators, as those, for example, which are relevant to the structure function $G_2$.

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