

Flow of entropy in the evolution of the $B^0 - \bar{B}^0$ system: Upper bound on CP violation from unidirectionality

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We have previously studied [1,6] the time dependence of a $B^0 - \bar{B}^0$ mixture in terms of its density matrix $\rho(t) = N(t)(\mathbb{1} + \vec{\zeta}(t) \cdot \vec{\sigma})/2$. The requirement that $\zeta(t)$, the absolute value of the Stokes vector $\vec{\zeta}(t)$, should evolve monotonically from its initial value $\zeta(0) = 0$ to its final value $\zeta(\infty) = 1$ was shown to lead to an upper bound on the CP violating overlap $\delta = \langle B_L | B_S \rangle$. In the present paper, we consider the entropy variable $S = -\text{tr}(\tilde{\rho} \log_2(\tilde{\rho}))$, where $\tilde{\rho} = \rho/N$, as an alternative measure of mixing. We show that exactly the same upper bound emerges from the requirement that the flow of entropy is unidirectional ($dS/dt < 0$). We compare the entropic current dS/dt with and without CP violation and identify certain physical features that appear when the bound on δ is violated.

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I. EVOLUTION OF $B^0 - \bar{B}^0$ IN TERMS OF THE STOKES VECTOR

In a previous paper [1] we studied the manner in which a $B^0 - \bar{B}^0$ state, prepared as an equal incoherent mixture, evolves into a final coherent (pure) state representing the long-lived neutral meson B_L . The analysis was done in terms of the 2×2 density matrix

$$\rho(t) = \frac{1}{2} N(t) [\mathbb{1} + \vec{\zeta}(t) \cdot \vec{\sigma}], \quad (1)$$

with initial values $N(0) = 1$, $\zeta(0) = |\vec{\zeta}(0)| = 0$. Explicit solution of the Schrödinger equation for $|B^0\rangle$ and $|\bar{B}^0\rangle$ yields the result

$$N(t) = \frac{1}{2(1 - \delta^2)} [e^{-t} + e^{-rt} - 2\delta^2 e^{-\frac{1}{2}(1+r)t} \cos \mu t], \quad (2)$$

and

$$\zeta(t) = \left[1 - \frac{1}{N(t)^2} e^{-(1+r)t} \right]^{\frac{1}{2}}. \quad (3)$$

Here, t is the proper time measured in units of τ_S , the lifetime of the short-lived eigenstate. In addition $r = \gamma_L/\gamma_S$, $\mu = \Delta m/\gamma_S$ with $\Delta m = m_L - m_S$. The parameter δ is the CP violating overlap of the two eigenstates

$$\delta = \langle B_L | B_S \rangle. \quad (4)$$

It was shown that $\zeta(t)$, the magnitude of the Stokes vector, undergoes a transition from monotonic to nonmonotonic behavior at a critical value

$$\delta_{\text{crit}} = \sqrt{\frac{1}{2} \left(\frac{1-r}{\mu} \right) \sinh \left(\frac{3\pi}{4} \frac{(1-r)}{\mu} \right)} \approx \sqrt{\frac{3\pi}{8} \frac{1-r}{\mu}}. \quad (5)$$

For the $B^0 - \bar{B}^0$ system, the theoretically expected value of $\Delta\gamma = \gamma_S - \gamma_L$ is $0.087 \pm 0.021 \text{ ps}^{-1}$ [2], compatible with the experimental value $0.100 \pm 0.013 \text{ ps}^{-1}$ as evaluated by the HFAG group [3]. From the same source we get $\Delta m = 17.69 \pm 0.08 \text{ ps}^{-1}$. With $r = 0.85 \pm 0.03$ and $\mu = 26.4 \pm 0.18$ the critical value of the CP violating parameter in (5) is $\delta_{\text{crit}} = (0.62 \pm 0.02)\%$. This is nearly 3 orders of magnitude higher than the expected value $\delta \approx 1.0 \times 10^{-5}$ in the standard model [4]. Surprisingly, an empirical determination of δ for the $B^0 - \bar{B}^0$ system, based on an asymmetry between $\mu^+ \mu^+$ and $\mu^- \mu^-$ events in the D0 experiment [5] had given a value incompatible with the standard model at the 2σ level. The recent evaluation of all experimental results by the HFAG group [3] yields an average $\delta_{\text{exp}} = (0.52 \pm 0.32)\%$ now compatible with the standard model prediction and the limit obtained from (5). The unexpected D0 result and the intensive discussion in the literature about its consequences, see e.g., Refs. [6,7], induced us to examine further the implication of the phase transition pointed out in Ref. [1].

II. EVOLUTION OF $B^0 - \bar{B}^0$ IN TERMS OF ENTROPY

To obtain further insight into the nature of the phase transition in $\zeta(t)$, we introduce a new variable $S(t)$ which is connected to the Stokes vector via

$$S(t) = -\frac{1 + \zeta(t)}{2} \log_2 \left(\frac{1 + \zeta(t)}{2} \right) - \frac{1 - \zeta(t)}{2} \log_2 \left(\frac{1 - \zeta(t)}{2} \right). \quad (6)$$

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This may be written formally as

$$S = -\text{tr}(\tilde{\rho} \log_2(\tilde{\rho})) = -\tilde{\lambda}_1 \log_2(\tilde{\lambda}_1) - \tilde{\lambda}_2 \log_2(\tilde{\lambda}_2), \quad (7)$$

where $\tilde{\rho} = \rho/N$ and $\tilde{\lambda}_1, \tilde{\lambda}_2$ are the eigenvalues of $\tilde{\rho}$, given explicitly by

$$\tilde{\lambda}_1 = \frac{1}{2}(1 + \zeta), \quad \tilde{\lambda}_2 = \frac{1}{2}(1 - \zeta). \quad (8)$$

The function $S(t)$ is closely related to the traditional von Neumann entropy $S_{\text{vN}} = -\text{tr}(\rho \log_2(\rho))$ defined for stable systems with $N(t) = \text{tr}(\rho) = 1$. The definition (7) takes into account that for a decaying $B^0 - \bar{B}^0$ system the norm $N(t)$ is different from unity.

The entropy as defined in (7) has the following desirable properties:

- (i) $S(t)$ lies between 0 and 1 for all δ, r, μ, t .
- (ii) For $\zeta = 0$ (incoherent limit), $S = 1$ (maximum disorder). For $\zeta = 1$ (coherent limit), $S = 0$ (minimum disorder).
- (iii) $S(t)$ has the important feature that the derivative is related to the derivative of $\zeta(t)$ by

$$\frac{dS}{dt} = -\frac{1}{2} \log_2 \left(\frac{1 + \zeta}{1 - \zeta} \right) \frac{d\zeta}{dt}. \quad (9)$$

This shows that the zeros of dS/dt are the same as the zeros of $d\zeta/dt$, implying that the critical value of the CP violating parameter at which dS/dt and $d\zeta/dt$ change sign is the same.

III. ENTROPIC CURRENT IN THE ABSENCE OF CP VIOLATION

We first consider the entropy function $S(t) = S_0(t)$ when there is no CP violation ($\delta = 0$). In this limit the modulus of the Stokes vector to be inserted in (6) is given by

$$\zeta_0(t) = \frac{e^{-t} - e^{-rt}}{e^{-t} + e^{-rt}}. \quad (10)$$

An important characteristic is the derivative dS_0/dt which is given by

$$\frac{dS_0}{dt} = \frac{-b^2 t e^{-bt}}{\ln(2)(1 + e^{-bt})^2}. \quad (11)$$

This function is entirely determined by the parameter $b = 1 - r$, with no dependence on μ . It is plotted in Fig. 1 behaving as $-t \exp(-bt)$ at large t and $-b^2 t / (4 \ln 2)$ at small t . The negative sign reflects the fact that S_0 is a decreasing function of time for all t . The decrease of entropy in the present situation contrasts with the increase of entropy familiar from stable thermodynamical systems. It is a consequence of the fact that a mixed $B^0 - \bar{B}^0$ state with arbitrary $S_0(0) > 0$ must ultimately reduce to the pure long-lived state B_L when the short-lived component has died out, implying that $S_0(\infty) = 0$. An interesting feature is the location of the minimum which is calculated from the root of the transcendental equation

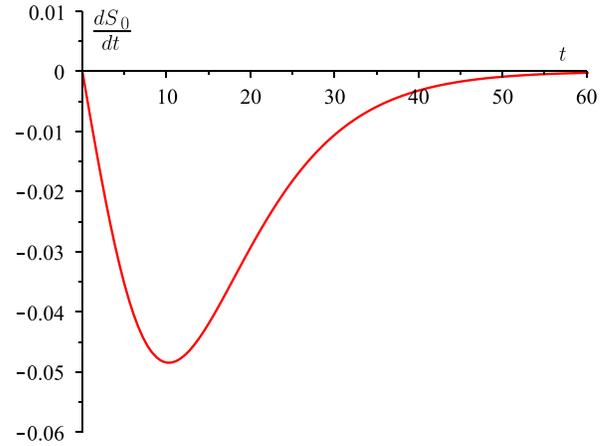


FIG. 1 (color online). dS_0/dt , the entropic current for a $(B_s^0 - \bar{B}_s^0)$ -like system in the tranquil state ($\delta = 0$) versus t in units of τ_S , the lifetime of B_s^0 . The area under the curve is -1 .

$$e^{-bt} = \frac{bt - 1}{bt + 1}. \quad (12)$$

The numerical result is

$$t = \frac{1.543\,405}{1 - r}. \quad (13)$$

We will refer to the curve dS_0/dt as the “entropic current in the tranquil state” ($\delta = 0$).

IV. IMPACT OF CP VIOLATION ON ENTROPIC CURRENT

When CP violation is taken into account ($\delta \neq 0$) the modulus of the Stokes vector changes to

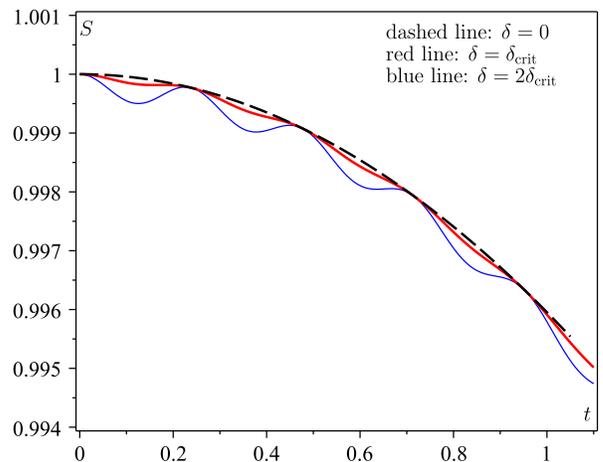


FIG. 2 (color online). Entropy function $S(t)$ versus $t \leq 1$ for a $(B_s^0 - \bar{B}_s^0)$ -like system using $\delta = 0, \delta_{\text{crit}}, 2\delta_{\text{crit}}$. For large values of t the oscillations are strongly damped. The curve for $\delta = \delta_{\text{crit}}$ defines the boundary between monotonic and nonmonotonic behavior of $\zeta(t)$.

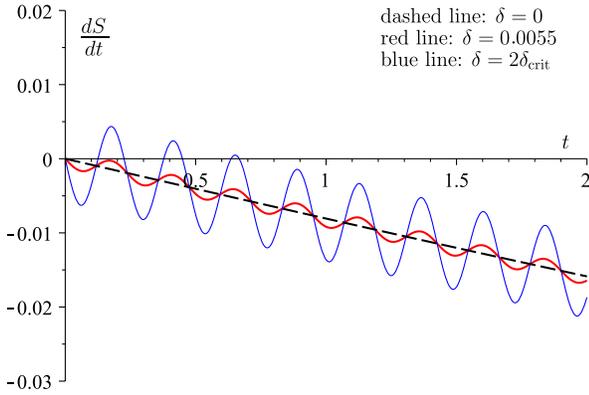


FIG. 3 (color online). Entropic current $dS(t)/dt$ for a $(B_s^0 - \bar{B}_s^0)$ -like system versus $t \leq 2$ for $\delta = 0$ (straight line), $\delta < \delta_{\text{crit}}$ (ripples) and $\delta = 2\delta_{\text{crit}}$ (eddies).

$$\zeta(t) = \sqrt{1 - \frac{e^{-(1+r)t}}{N(t)^2}}, \quad (14)$$

where $N(t)$ is given in Eq. (2). Inserting this $\zeta(t)$ in (6) yields $S(t)$ as shown in Fig. 2 for $\delta = \delta_{\text{crit}}$ and $\delta = 2\delta_{\text{crit}}$. For comparison $S_0(t)$ is also included.

The impact of CP violation is more clearly seen in the entropic current dS/dt and its comparison with dS_0/dt . This comparison is shown in Fig. 3. The effect of CP violation is essentially a modulation of the curve dS_0/dt by oscillations. The curve dS_0/dt is, in the interval $0 < t < 2$, practically a straight line. The modulation consists of an oscillating function, with frequency Δm and amplitude proportional to δ^2 . For $\delta < \delta_{\text{crit}}$ the amplitude of the oscillations (“ripples”) is small enough that the entropic current remains negative in sign ($dS/dt < 0$). For $\delta > \delta_{\text{crit}}$ the oscillations grow in amplitude to the extent that some of the ripples become “eddies”: these are regions in which the entropic current reverses its sign. This fact is illustrated

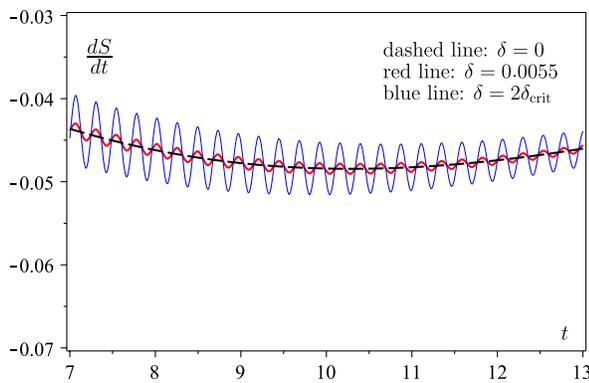


FIG. 4 (color online). Entropic current $dS(t)/dt$ for a $(B_s^0 - \bar{B}_s^0)$ -like system versus t in the neighborhood of the minimum of Fig. 1 for $\delta = 0$ (dashed curve), showing small ripples for $\delta < \delta_{\text{crit}}$ and larger ripples for $\delta = 2\delta_{\text{crit}}$.

in Fig. 3 with the help of two curves showing the entropic current for a value $\delta = 0.0055$ slightly below δ_{crit} and $\delta = 2\delta_{\text{crit}}$ respectively. The transition in the sign of dS/dt as δ crosses the critical value δ_{crit} is equivalent to the transition from monotonic to nonmonotonic behavior of the Stokes vector $\zeta(t)$ noted in Ref. [1].

For completeness the effect of CP violation at larger values of t especially in the neighborhood of the minimum of Fig. 1 is shown in Fig. 4.

V. REMARKS

- (i) Our purpose in this paper has been to elucidate the phase transition in the evolution of a $B_0 - \bar{B}_0$ mixture, pointed out in Ref. [1], by going from a Stokes vector description to one involving entropy. There is a change in sign in the entropic current dS/dt at exactly the same critical value of the CP violating parameter δ , written explicitly in Eq. (5). Examination of the entropy flow below and above the critical value (Fig. 3) shows that the loss of unidirectionality of the entropic flow is accompanied by the appearance of one or more eddylike transients that cause a reversal in the sign of dS/dt in limited intervals of time.
- (ii) The onset of CP violation causes, in the subcritical domain, a ripplelike perturbation in dS/dt , with an amplitude proportional to δ^2 , which is not large enough to change the negative sign of the entropic current. For $\delta > \delta_{\text{crit}}$ these ripples grow in size to the extent that the entropic current penetrates into the region $dS/dt > 0$. When that happens, the arrow of time reverses direction in certain intervals of time, compared to its direction in the tranquil state.
- (iii) Our study has focused on a simple two-level system whose eigenstates have different masses and lifetimes. In the absence of CP violation, the quantum mechanical entropy of such an unstable (open) system decreases with time. We have found that this behavior is affected by CP violation. In a recent paper [8] the question has been raised whether the monotonic increase of entropy in the macroscopic universe (the progression from order to disorder) can also be influenced by CP violation. Examples are given where CP or T violation can reverse this evolution. In a different context [9] it has been argued that the unidirectionality of time could itself be a consequence of CP violation. Our analysis of the $B_0 - \bar{B}_0$ system shows that questions concerning entropy and the impact of T violation on the arrow of time can be discussed in a meaningful way also in the context of pristine two-level systems such as $K_0 - \bar{K}_0$ or $B_0 - \bar{B}_0$, which have given us profound insights into the nature of the world under C , P , and T transformations.

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